Topological constraints generating structures

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Self-organization without ‘blueprints’

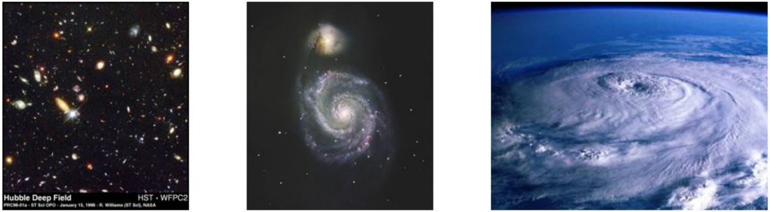
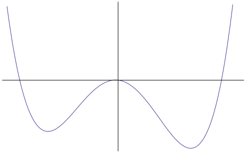


Figure: Self-organization in physical systems: Vortexes are chiral structures spontaneously created without *programs*.

Basic mechanisms of self-organization



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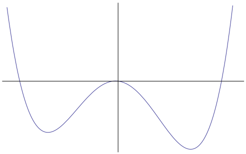

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- Nontrivial 'space': Topological constraints $\rightarrow$ 'effective energy'

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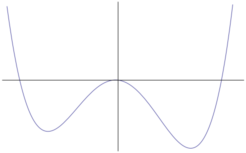
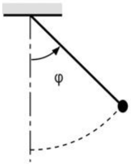


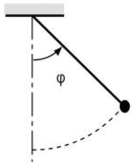
- Nontrivial 'energy' $\rightarrow$ nontrivial structure
- Nontrivial 'space': Topological constraints $\rightarrow$ 'effective energy'
- Casimir invariants $\rightarrow$ foliation of phase space

The simplest example of micro=canonical mechanics: pendulum



(a)

The simplest example of micro=canonical mechanics: pendulum



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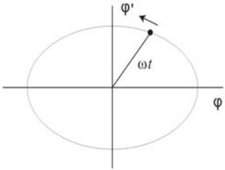
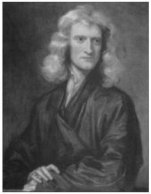


(b)



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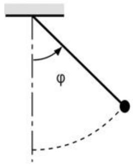

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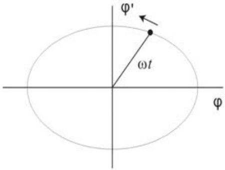

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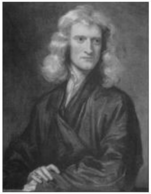
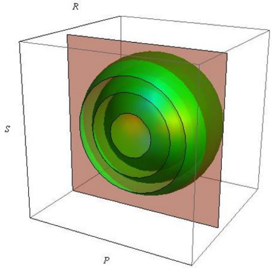
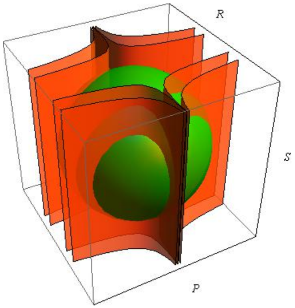
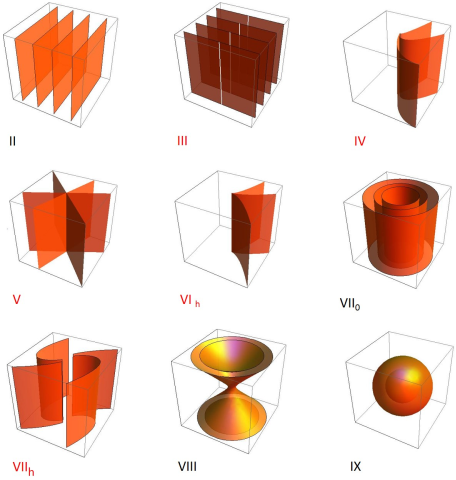




Other possibilities of spaces

Bianchi classification of 3D Lie algebras





Foliated phase space →

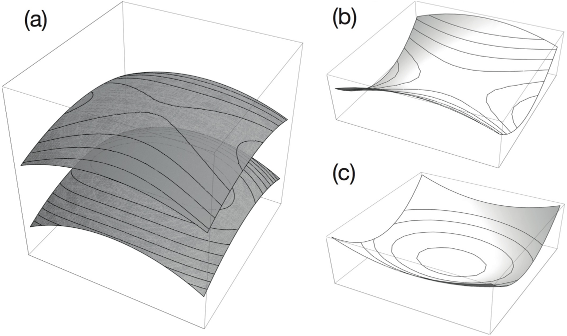


Figure: The energy (Hamiltonian) may have a nontrivial distribution on each leaf of the foliated phase space.

Basic Formulation

- Hamiltonian system:
$$\partial_t u = J\partial_u H$$
with state vector $u \in X$ (phase space), Poisson operator J defining a Poisson bracket
$$\{F, G\} = \langle \partial_u F, J\partial_u G \rangle,$$
and a Hamiltonian $H \in C^\infty_{\{\ ,\ \}}(X)$ (Poisson algebra).
- The adjoint representation:
$$\frac{d}{dt}F = \{F, H\} = -\mathrm{ad}_H F.$$
- Canonical system: $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \to$ symplectic geometry
- Noncanonical system has topological defects: $\mathrm{Ker}(J) = \mathrm{Coker}(J)$.
- Casimir invariant: $J\partial_u C = 0$, i.e., $\mathrm{Ker}(J) \ni v = \partial_u C$.

The origin of Casimir invariant (a tutorial example) (1)

- Let us start with a 6-dimensional phase space:

$$\boldsymbol{z} := (q_1, q_2, q_3, p_1, p_2, p_3)^T \in X_{\boldsymbol{z}} = \mathbb{R}^6, \tag{1}$$

on which we define a canonical Poisson bracket

$$\{F, G\}_{\boldsymbol{z}} := (\partial_{\boldsymbol{z}} F, J_{\boldsymbol{z}} \partial_{\boldsymbol{z}} G) \tag{2}$$

$$J_{\boldsymbol{z}} = J_c := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{3}$$

- Denoting $\boldsymbol{q} = (q_1, q_2, q_3)^T$ and $\boldsymbol{p} = (p_1, p_2, p_3)^T$, we define

$$\boldsymbol{\omega} := \boldsymbol{q} \times \boldsymbol{p} \in X_{\boldsymbol{\omega}}. \tag{4}$$

We *reduce* $C^\infty(X_{\boldsymbol{z}})$ to $C^\infty(X_{\boldsymbol{\omega}})$:

$$\begin{aligned} \{F(\boldsymbol{q} \times \boldsymbol{p}), G(\boldsymbol{q} \times \boldsymbol{p})\}_{\boldsymbol{z}} &= \{F(\boldsymbol{\omega}), G(\boldsymbol{\omega})\}_{\boldsymbol{\omega}} \\ &:= (\partial_{\boldsymbol{\omega}} F, (-\boldsymbol{\omega}) \times \partial_{\boldsymbol{\omega}} G). \end{aligned} \tag{5}$$

The origin of Casimir invariant (a tutorial example) (2)

- Denoting

$$J_\omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{6}$$

we may write

$$\{F(\boldsymbol{\omega}), G(\boldsymbol{\omega})\}_\omega := (\partial_\omega F, J_\omega \partial_\omega G), \tag{7}$$

which is the $\mathfrak{so}(3)$ Lie-Poisson bracket.

- The reduced Poisson algebra, to be denoted by $C^\infty_{\{\ ,\ \}_\omega}(X_\omega)$, is noncanonical, having a Casimir invariant

$$C = \frac{1}{2}|\boldsymbol{\omega}|^2. \tag{8}$$

- Physically, X_ω is the phase space of a rigid-body on an inertia frame co-moving with the center of mass. The mechanical degree of freedom is, then, only the angular momentum $\boldsymbol{\omega}$; the phase space X_ω may be identified with $\mathfrak{so}(3)$.

- Vortex dynamics is as an infinite-dimensional generalization of the aforementioned $\mathfrak{so}(3)$ noncanonical Lie-Poisson system.
- Let $\boldsymbol{Z} = (Q(\boldsymbol{x}), P(\boldsymbol{x}))^T \in X_{\boldsymbol{Z}}$ be a 2-component field on a base manifold T^2, on which we define a canonical Poisson bracket

$$\{F, G\}_{\boldsymbol{Z}} := (\partial_{\boldsymbol{Z}} F, J_c \partial_{\boldsymbol{Z}} G), \quad J_c := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- We define

$$\boldsymbol{\omega} = [Q, P] := \mathrm{d}Q \wedge \mathrm{d}P \in X_\omega,$$

and reduce $C^\infty(X_{\boldsymbol{Z}})$ to $C^\infty(X_{\boldsymbol{\omega}})$:

$$\begin{aligned} \{F([Q,P]), G([Q,P])\}_{\boldsymbol{Z}} \{F(\omega), G(\omega)\}_\omega &= (\partial_\omega F, [\omega, \partial_\omega G]) \\ &:= (\partial_\omega F, J_\omega \partial_\omega G). \end{aligned}$$

Vortex dynamics (2)
Hamiltonian and Casimir

- We write the vorticity $\omega = -\Delta\varphi$ with the Gauss potential φ (Δ is the Laplacian). Then, $\boldsymbol{V} = {}^t(\partial_y\varphi, -\partial_x\varphi)$.
- Given a Hamiltonian

$$H_{\mathrm{E}}(\omega) = -\frac{1}{2}\int \omega \cdot (\Delta^{-1}\omega)\mathrm{d}^2x,$$

Hamilton's equation $\partial_t\omega = J\partial_\omega H_{\mathrm{E}}$ reproduces the vortex equation for Eulerian flow,

$$\partial_t\omega + \boldsymbol{V} \cdot \nabla\omega = 0.$$

- The reduced bracket $\{\ ,\ \}_\omega$ has a Casimir = enstrophy:

$$C = \int f(\omega)\ \mathrm{d}^2x.$$

Topological constraints in ideal MHD (1)

Naïve formulation

- We consider a barotropic MHD:

$$\partial_t \rho = -\nabla\cdot(\boldsymbol{V}\rho),$$
$$\partial_t \boldsymbol{V} = -(\nabla\times\boldsymbol{V})\times\boldsymbol{V} - \nabla(h + V^2/2) + \rho^{-1}(\nabla\times\boldsymbol{B})\times\boldsymbol{B},$$
$$\partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{V}\times\boldsymbol{B}).$$

- The local magnetic flux on an arbitrary co-moving surface $\sigma(t)$

$$\Phi_\sigma(t) = \int_{\sigma(t)} \boldsymbol{\nu}\cdot\boldsymbol{B}\,\mathrm{d}^2x = \oint_{\partial\sigma(t)} \boldsymbol{\tau}\cdot\boldsymbol{A}\,\mathrm{d}x$$

is a constant of motion.

- Because of this infinite set of conservation laws, the magnetic field lines are forbidden to change their topology.

The phase space X contains the state vector $u = (\rho, \boldsymbol{V}, \boldsymbol{B})^T$, and the Hamiltonian and Poisson operator are given as follows:

$$H = \int_\Omega \left\{ \rho \left[\frac{V^2}{2} + \mathcal{E}(\rho) \right] + \frac{B^2}{2} \right\} \mathrm{d}^3 x, \qquad (9)$$

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -\rho^{-1}(\nabla\times\boldsymbol{V})\times & \rho^{-1}(\nabla\times\circ)\times\boldsymbol{B} \\ 0 & \nabla\times\left(\circ\times\rho^{-1}\boldsymbol{B}\right) & 0 \end{pmatrix}. \qquad (10)$$

The Poisson operator $\mathcal{J}$ has well-known Casimir invariants:

$$C_1 = \int_\Omega \rho \, \mathrm{d}^3 x,$$

$$C_2 = \frac{1}{2}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B} \, \mathrm{d}^3 x,$$

$$C_3 = \int_\Omega \boldsymbol{V}\cdot\boldsymbol{B} \, \mathrm{d}^3 x,$$

Topological constraints in ideal MHD (3)

Structures created on Casimir leaves: Beltrami fields (Taylor relaxed states:)

- The magnetic helicity C_2 is a *robust* constant; hence the plasma relaxes into the minimum energy state under the constraint on C_2.
- The minimizer is called the *Taylor relaxed state*.
- The variational principle of minimizing $H_\mu := H - \mu C_3$ yields

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{B}.$$

For the existence of solutions, see Y & Giga,, Math. Z. **204** (1990), 235-245.

- Let u_μ be the equilibrium point of H_μ, and approximate $H_\mu(u_\mu + \tilde{u}) \approx H_\mu(u_\mu) + \frac{1}{2}\langle \mathcal{H}_\mu \tilde{u}, \tilde{u}\rangle$ with a linear operator $\mathcal{H}_\mu$. If

$$c\|\tilde{u}\|^2 \leq \langle \mathcal{H}_\mu \tilde{u}, \tilde{u}\rangle$$

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- Problem: What is $\mathcal{H}_\mu$? What is the condition on μ for the stability?

Topological constraints in ideal MHD (3-2)

Structures created on Casimir leaves: Beltrami fields (Taylor relaxed states:)

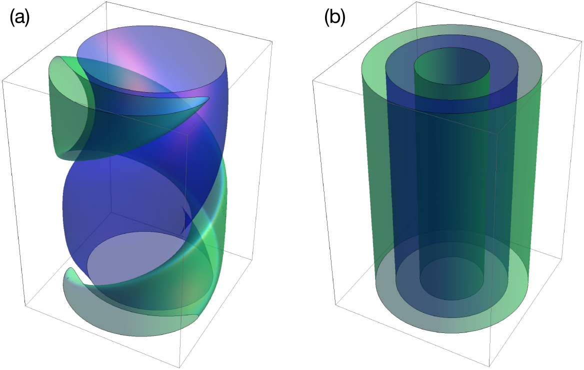

Figure: Bifurcation of Taylor relaxed states (the figures plot the magnetic surfaces). The helicity C_2 plays the role of a bifurcation parameter.

Y & R. L. Dewar; J. Phys. A: Math. Theor. **45** (2012), 365502 (36pp).

Topological constraints in ideal MHD (4)

Structures created on Casimir leaves: Alfvén waves

- The stationary point of $H - \mu C_1 - \lambda C_3$ is given by $\rho \boldsymbol{V} = \lambda \boldsymbol{B}$, $\boldsymbol{B} = \lambda \boldsymbol{V}$, and Bernoulli's relation $\rho V^2/2 + h(\rho) = \mu$.
- Nontrivial solutions are given by $\lambda^2 = 1$, $\rho = 1$, and

$$\boldsymbol{V} = \pm \boldsymbol{B}. \quad |\boldsymbol{V}| = \mathrm{constant}. \tag{11}$$

- Putting (with $\boldsymbol{B}_0 =$ constant)

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}, \quad \boldsymbol{V} = \pm \boldsymbol{B}_0 + \boldsymbol{v},$$

  and boosting $\boldsymbol{x} \to \boldsymbol{x} \mp \boldsymbol{B}_0 t$, the *wave component* satisfies

$$\begin{aligned} \partial_t \boldsymbol{v} &= -(\nabla \times \boldsymbol{v}) \times \boldsymbol{v} + (\nabla \times \boldsymbol{b}) \times (\boldsymbol{b} + \boldsymbol{B}_0) - \nabla(V^2/2 + h), \\ \partial_t \boldsymbol{b} &= \nabla \times [\boldsymbol{v} \times (\boldsymbol{b} + \boldsymbol{B}_0)], \end{aligned}$$

- The determining equations (11) have a large set of exact nonlinear solutions, implying that Alfvén waves, propagating on a homogeneous ambient field, have a large degree of freedom; arbitrarily shaped waves propagate keeping the wave forms.

Conclusion

1. Emergence of nontrivial structures
    $\leftarrow$ Nontrivial 'effective energy'
        $\leftarrow$ Topological constraints $=$ Phase space foliation

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Questions:

1. What is the 'origin' of the Casimirs?
2. How about the local constraints?

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- Theoretically, a constant of motion is a reflection of some *symmetry*.

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The origin of Casimirs

- Theoretically, a constant of motion is a reflection of some *symmetry*.
- What is the *symmetry* that generates a Casimir?
- The constancy of the Casimir is independent of the Hamiltonian. Hence, it is conceivable that the Casimir pertains to some *gauge symmetry*.
- There must be some fundamental canonical system beneath the MHD system, and the MHD system is its *reduction*. The Casimir is, then, the Noether charge of the gauge transformation pertinent to the redundancy.

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- Clebsch field $[p_j(\boldsymbol{x}), q_k(\boldsymbol{y})] = \delta_{jk}\delta(\boldsymbol{x} - \boldsymbol{y})$

$$\Rightarrow \begin{cases} \{F, G\}_f = (\partial_{\boldsymbol{u}} F, J\partial_{\boldsymbol{u}} G): \text{ Classical fluid systems} \\ [\psi_j(\boldsymbol{x}), \psi_k^*(\boldsymbol{y})] = \frac{1}{i\hbar}\delta_{jk}\delta(\boldsymbol{x} - \boldsymbol{y}): \text{ Quantum field systems} \end{cases}$$

Classical realization as an Eulerian flow (I)

- Phase space $\boldsymbol{Z}(\boldsymbol{x}) = (q_0(\boldsymbol{x}), q_1(\boldsymbol{x}), \cdots, p_0(\boldsymbol{x}), p_1(\boldsymbol{x}), \cdots) \in X$.
- Canonical Poisson algebra on $C^\infty(X)$ with

$$\{F, G\}_c = (\boldsymbol{Z}, [\partial_{\boldsymbol{Z}} F, \partial_{\boldsymbol{Z}} G]) = (\partial_{\boldsymbol{Z}} F, J_c \partial_{\boldsymbol{Z}} G), \quad J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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- The *Clebsch reduction* to the *fluid variables*:

$$\boldsymbol{u} = (\rho, \boldsymbol{V}), \quad \begin{cases} \rho = p_0 \\ \boldsymbol{V} = \frac{1}{p_0}(p_0 \mathrm{d}q_0 + p_1 \mathrm{d}q_1 + \cdots) \end{cases}$$

yields a non-canonical Poisson system with

$$\{F(\boldsymbol{u}(\boldsymbol{Z})), G(\boldsymbol{u}(\boldsymbol{Z}))\}_c = \{F(\boldsymbol{u}), G(\boldsymbol{u})\}_f = (\partial_{\boldsymbol{Z}} F, J_f \partial_{\boldsymbol{Z}} G),$$

$$J_f = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{V})\times \end{pmatrix}.$$

Classical realization as an Eulerian flow (II)

- With a Hamiltonian (conventional fluid energy)

$$\mathscr{H}_c = \int \left[\frac{|\boldsymbol{V}|^2}{2} + U(\rho) \right] \rho \, dx,$$

the Hamilton's equation $\dot{\boldsymbol{u}} = J_f \partial_{\boldsymbol{u}} H_c$ reads as the classical system of ideal fluid:

$$\partial_t \rho + \nabla \cdot (\boldsymbol{V}\rho) = 0,$$
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- The noncanonical bracket $\{\ ,\ \}_f$ has Casimirs:

$$C_1 = \int_\Omega \rho \, \mathrm{d}^3 x,$$
$$C_2 = \frac{1}{2} \int_\Omega \boldsymbol{V} \cdot (\nabla \times \boldsymbol{V}) \, \mathrm{d}^3 x.$$

Classical realization as an MHD plasma (I)

- The phase space:

$$\boldsymbol{Z}(\boldsymbol{x}) = (q_0, q_1, \cdots, s_1, \cdots; p_0, p_1, \cdots, r_1, \cdots) \in X.$$

- Canonical Poisson algebra with $\{F, G\}_c = (\boldsymbol{Z}, [\partial_{\boldsymbol{Z}} F, \partial_{\boldsymbol{Z}} G])$.

Classical realization as an MHD plasma (I)

- The phase space:

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- Canonical Poisson algebra with $\{F, G\}_c = (\boldsymbol{Z}, [\partial_{\boldsymbol{Z}} F, \partial_{\boldsymbol{Z}} G])$.
- The *Clebsch reduction* to the *MHD field*:

$$\boldsymbol{u} = (\rho, \boldsymbol{V}, \boldsymbol{B}), \quad \begin{cases} \rho = p_0 \\ \boldsymbol{V} = \frac{1}{p_0}[p_0 \mathrm{d}q_0 + p_1 \mathrm{d}q_1 + \cdots - (r_1 \mathrm{d}s_1 \cdots)] \\ \boldsymbol{B} = \mathrm{d}\left(\frac{p_1}{p_0}\right) \wedge \mathrm{d}s_1 + \cdots \end{cases}$$

yields a non-canonical Poisson system with

$$\{F(\boldsymbol{u}(\boldsymbol{Z})), G(\boldsymbol{u}(\boldsymbol{Z}))\}_c = \{F(\boldsymbol{u}), G(\boldsymbol{u})\}_f = (\partial_{\boldsymbol{Z}} F, J_{MHD} \partial_{\boldsymbol{Z}} G),$$

$$J_{MHD} = \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{V})\times & \rho^{-1}(\nabla \times \circ) \times \boldsymbol{B} \\ 0 & \nabla \times (\circ \times \rho^{-1}\boldsymbol{B}) & 0 \end{pmatrix}.$$

To formulate the local magnetic flux as a Casimir invariant, we extend the phase space in order to include topological indexes information in the set of dynamical variables.

Adding a 2-form $\check{\boldsymbol{B}}$, which we call a *phantom field*, to the MHD variables, gives the extended phase space state vector

$$\tilde{u} = (\rho, \boldsymbol{V}, \boldsymbol{B}, \check{\boldsymbol{B}})^T, \tag{12}$$

on which we define a degenerate Poisson manifold by

$$\tilde{\mathcal{J}} = \left(\begin{array}{cccc} 0 & -\nabla\cdot & 0 & 0 \\ -\nabla & -\rho^{-1}(\nabla\times\boldsymbol{V})\times & \rho^{-1}(\nabla\times\circ)\times\boldsymbol{B} & \rho^{-1}(\nabla\times\circ)\times\check{\boldsymbol{B}} \\ 0 & \nabla\times\left(\circ\times\rho^{-1}\boldsymbol{B}\right) & 0 & 0 \\ \hline 0 & \nabla\times\left(\circ\times\rho^{-1}\check{\boldsymbol{B}}\right) & 0 & 0 \end{array} \right). \tag{13}$$

Using the same Hamiltonian (9), we obtain an extended dynamics governed by exactly the same MHD equations together with an additional equation $\partial_t \check{\boldsymbol{B}} = \nabla\times(\boldsymbol{V}\times\check{\boldsymbol{B}})$.

Topological constraints in ideal MHD (continued)

Circulation theorem represented by the Casimirs of the extended system

- The extended Poisson operator (13) has the set of Casimir invariants composed of C_1, C_2, and a new *cross helicity*

$$C_4 = \int_\Omega \boldsymbol{A} \cdot \check{\boldsymbol{B}} \, \mathrm{d}^3 x,$$

as well as a phantom magnetic helicity

$$C_5 = \frac{1}{2} \int_\Omega \check{\boldsymbol{A}} \cdot \check{\boldsymbol{B}} \, \mathrm{d}^3 x.$$

- Interestingly, the original (standard) cross helicity $C_3 = \int_\Omega \boldsymbol{V} \cdot \boldsymbol{B} \, \mathrm{d}^3 x$ is no longer a Casimir invariant of the extended system, although it is still a constant of motion. The constancy of C_3 is now due to the "symmetry" of a Hamiltonian with ignorable dependence on the phantom field $\check{\boldsymbol{B}}$;

Conclusion (2)

1. Beneath various fluid systems (e.g. Eulerian fluid, MHD, etc.) is the canonical Lie-Poisson system of classical fields which is a realization of the Heisenberg algebra.
2. Variety of *foliated phase spaces*, on which nontrivial effective energies generate interesting structures, are derived by the *Clebsch reduction*.
3. The Casimir invariants of the fluid systems are the Noether charges; the co-adjoint group action generated by the Casimir represents the gauge symmetry.

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