Mathematical analysis of pulsatile flow and vortex breakdown

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Pulsatile flow problem

Stability and instability of a time-periodic Navier-Stokes flow (or real flow)

- The key in physics: Womersley number

- A pipe $\Omega_{\mathcal{R}}$ as $\Omega_{\mathcal{R}} := \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < \mathcal{R},\ 0 < x_3 < \ell\}$
- with its side-boundary $\partial\Omega_{\mathcal{R}} = \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = \mathcal{R},\ 0 < x_3 < \ell\}$.

The incompressible Navier-Stokes equations are described as follows:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_{\mathcal{R}}, \quad u = 0 \quad \text{on} \quad \partial\Omega_{\mathcal{R}}$$

with $u = u(x,t) = (u_1(x_1,x_2,x_3,t), u_2(x_1,x_2,x_3,t), u_3(x_1,x_2,x_3,t))$ and $p = p(x,t)$.

Recent research of the pulsatile flow

Trip-Kuik-Westerweel-Poelma (2012)

If p_1 and p_2 are the pressure at the ends of the pipe $\Omega_{\mathcal{R}}$, the pressure gradient can be expressed as $(p_1 - p_2)/\ell$.

- If the pressure gradient is time-independent, $(p_1 - p_2)/\ell =: p_s$, then we can find the stationary Navier-Stokes flow (Poiseuille flow):

  $$u_s = (u_1, u_2, u_3) = (0, 0, \frac{p_s}{4\nu\ell}(\mathcal{R}^2 - r^2)),$$

  where $r = \sqrt{x_1^2 + x_2^2}$.
- The oscillating pressure gradient case,

  $$\frac{p_1(t) - p_2(t)}{\ell} = p_o e^{iNt}$$

  which is periodic in time. Then its corresponding solution u_o can be written explicitly by using a Bessel function with $u_1 = u_2 = 0$.

Womersley number α

$$\alpha = \mathcal{R}\sqrt{\frac{N}{\nu}}$$

Also they define

- oscillatory Reynolds number
- the mean Reynolds number (as usual one)

by using u_o and u_s respectively (in this case the main flow is $u_o + u_s$).
According to their experiment,

- measurement at different Womersley numbers yield similar transition behavior
- variation of the oscillatory Reynolds number also appear to have little effect to the transition behavior

Thus they conclude that the transition seems to be determined only by the mean Reynolds number.

However it seems they did not investigate the effect of the swirl component (azimuthal component)

Vortex breakdown

Brons-Voigt and Sorensen (1999) systematically determine the possible flow topologies of the steady axisymmetric Navier-Stokes flow in a cylindrical container (such as $\Omega_{\mathcal{R}}$) with rotating end-covers.

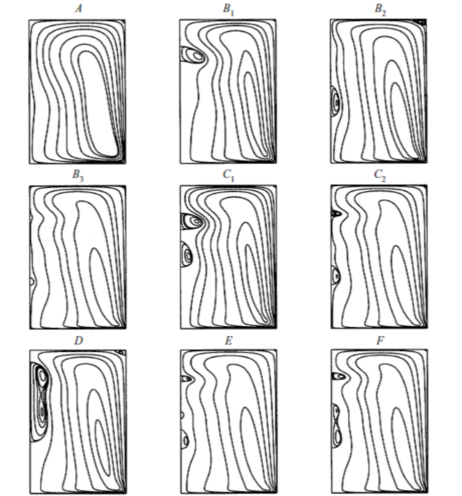

Figure: Brons-Voigt-Sorensen 1999 (c.f. Hsu-Notsu-Y JFM 2016)

Since we do not take the boundary layer into account, the high Reynolds number flow $\approx$ the Euler flow

Euler equations in a pipe $\Omega_{\mathcal{R}}$

$$\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_{\mathcal{R}},$$
$$u(x,t)|_{x_3=0} = (0, 0, U_{in}(r,t)) \quad \text{with} \quad U_{in} > 0,$$

Inflow conditions

- Pulsatile flow case: $U_{in} = U_s(r) + U_o(r) g(t)$

$$\sup_{0\le j\le 2} |\partial_r^j U_s(r)| + \sup_{0\le j\le 2} |\partial_r^j U_o(r)| \lesssim 1 \text{ with rapidly increasing } g.$$

- Vortex breakdown case: Not rapidly increasing inflow U_{in}:

$$\sup_{0\le j,k\le 2} |\partial_r^j \partial_t^k U_{in}(r,t)| \lesssim 1.$$

Restrict to the axisymmetric flow

- e_r: radial direction, e_θ: rotation direction, e_z: axial direction

$v_r = v_r(r,z,t)$, $v_\theta = v_\theta(r,z,t)$, $v_z = v_z(r,z,t)$ be such that

$$u = v_r e_r + v_\theta e_\theta + v_z e_z$$

Then v_r, v_z, v_θ satisfy the following:

Axisymmetric Euler equations (no pressure on e_θ-direction)

$$\begin{aligned}
\partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - \frac{v_\theta^2}{r} + \partial_r p &= 0, \\
\partial_t v_\theta + v_r \partial_r v_\theta + v_z \partial_z v_\theta + \frac{v_r v_\theta}{r} &= 0, \\
\partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z + \partial_z p &= 0, \\
\frac{\partial_r (r v_r)}{r} + \partial_z v_z &= 0.
\end{aligned}$$

The key definitions

We always assume the vector field u is unilateral, that is, $u \cdot e_z > 0$.

Axis-length streamline $\bar{\Phi}$ (definition)

For fixed $t > 0$, $\partial_z \bar{\Phi}(z) = (u/u \cdot e_z)\,(\bar{\Phi}(z), t)$ with

$$\bar{\Phi}(z) = (\bar{R}(z) \cos \bar{\Theta}(z), \bar{R}(z) \sin \bar{\Theta}(z), z),$$

$\bar{R}(z) = \bar{R}(\bar{r}_0, z, t)$, $\bar{R}(\bar{r}_0, 0, t) = \bar{r}_0$, $\bar{\Theta}(z) = \bar{\Theta}(z, t)$.

Rate of disturbing laminar profile (key definition)

$$\begin{aligned}
L^0(\bar{r}_0, z, t) &= |\partial_{\bar{r}_0} \bar{R}| + |\partial_r \bar{R}^{-1}| \\
L^x(\bar{r}_0, z, t) &:= \sum_{\substack{1 \le j+k \le 3, \\ (j,k) \neq (0,1)}} |\partial_z^j \partial_{\bar{r}_0}^k \bar{R}| + \sum_{\substack{1 \le j+k \le 3 \\ (j,k) \neq (0,1)}} |\partial_z^j \partial_r^k \bar{R}^{-1}| \\
L^t(\bar{r}_0, z, t) &= |\partial_t \bar{R}^{-1}| + |\partial_t^2 \bar{R}^{-1}| + |\partial_t \partial_{\bar{r}_0} \bar{R}| + |\partial_t \partial_{\bar{z}_0} \bar{R}|.
\end{aligned}$$

Remarks

- Minumum value of L^0 is 2, since $|\partial_r \bar{R}^{-1}| = 1/|\partial_{\bar{r}_0} \bar{R}|$.
- The typical Euler solution $u = (0, 0, g)$ is the typical laminar flow. In this case $L^0 \equiv 2, \quad L^x \equiv 0 \quad \text{and} \quad L^t \equiv 0 \quad \text{for any} \quad g$.

Pulsatile flow case (First theorem)

For any $x \in \Omega_{\mathcal{R}}$ satisfying $u_0(x) \cdot e_\theta \neq 0$, then there is a smooth function g such that $|g| \approx 1$, $g'(t) \to \infty$, $g''(t) \to \infty$ $(t \to t_b)$,

$$L^t(\bar{r}_0, z, t) \to \infty$$

for $t \to t_b$ (compare with the above remark).

Vortex breakdown case (Second theorem)

For any $\epsilon > 0$, there is $\delta > 0$ such that if $|\partial_r v_\theta(0)| > 1/\delta$ (this should be corresponding to rotating top and bottom boundaries), $|\partial_r \partial_z v_\theta(0)| \lesssim 1$, $|\partial_r^2 v_\theta(0)| \lesssim 1$, $|L^0| + |L^x| \lesssim 1$ on the axis,

then $|L^t| > 1/\epsilon$ on the axis.

In pure mathematics, these results are corresponding to illposedness.

Strategy

Assume $|L^t| \lesssim 1/\epsilon$ and employ a contradiction argument.

First, recover v_r and v_z by using $\bar{R}$ and U_{in}.
By the Gauss' divergence theorem,

We have the following formula of v_z and v_r:

$$v_z(r,z,t) = \rho(\bar{R}^{-1}(r,z,t),z,t)u_z(\bar{R}^{-1}(r,z,t),0,t) = \rho(\bar{R}^{-1},z,t)U_{in}$$

$$v_r(r,z,t) = (\partial_z \bar{R})(\bar{R}^{-1}(r,z,t),z,t)u_z(r,z,t).$$

$$\text{with} \quad \rho(\bar{r}_0,z,t) = \frac{2\bar{r}_0}{\partial_{\bar{r}_0}\bar{R}(\bar{r}_0,z,t)^2}.$$

We define the Lagrangian flow on the meridian plane (r,z-plane). Let

$$\frac{d}{dt}Z_*(t) = v_z(R_*(t), Z_*(t), t),$$
$$Z_*(0) = z_0$$

and

$$\frac{d}{dt}R_*(t) = v_r(R_*(t), Z_*(t), t),$$
$$R_*(0) = r_0$$

with $Z_*(t) = Z_*(r_0, z_0, t)$ and $R_*(t) = R_*(r_0, z_0, t)$.

translate the time t to the axis-length z

Since $v_z > 0$, then we can define the inverse of Z_* in t: $t = Z_{*t}^{-1}(z, r_0, z_0)$.

We can estimate the Lagrangian deformation (for fixed t)

$\partial_{z_0} Z_*$, $\partial_{z_0} R_*$, $\partial_{r_0} Z_*$, $\partial_{r_0} R_*$, $\partial_z Z_*^{-1}$, $\partial_z R_*^{-1}$, $\partial_r Z_*^{-1}$, $\partial_r R_*^{-1}$

We can recover v_θ by Z_* and R_*.

By the Euler equation of v_θ, we see that

$$\partial_t v_\theta(R_*(t), Z_*(t), t) \quad = \quad -\frac{v_r(R_*(t), Z_*(t), t) v_\theta(R_*(t), Z_*(t), t)}{R_*(t)}$$

Applying the Gronwall equality, we see

Formula of v_θ

$$v_\theta(r, z, t) = v_\theta(r_0, z_0, 0) \exp\left\{ -\int_0^t \frac{v_r(R_*(r_0, z_0, t'), Z_*(r_0, z_0, t'), t')}{R_*(r_0, z_0, t')} dt' \right\}$$

with $r_0 = R_*^{-1}(r, z, t)$ and $z_0 = Z_*^{-1}(r, z, t)$.

Lagrangian flow $\Phi_*(t)$ (definition)

$$\frac{d}{dt}\Phi_*(x,t) = u(\Phi_*(x,t),t), \quad \Phi_*(x,0) = x \in \Omega_{\mathcal{R}}.$$

Axis-length trajectory

Let Φ be such that

$$\Phi(z) := (R(z)\cos\Theta(z), R(z)\sin\Theta(z), z)$$

and we choose $R(z)$ and $\Theta(z)$ in order to satisfy $\Phi(z) = \Phi_*(x, Z_{*t}^{-1}(z))$

Remark: $R(z) = R_*(Z_{*t}^{-1}(z))$.

Connection between trajectory and streamline

$$R(z)\Theta'(z) = \frac{v_\theta(R(z), z, Z_{*t}^{-1}(z))}{v_z(R(z), z, Z_{*t}^{-1}(z))},$$

$$R'(z) = \frac{v_r(R(z), z, Z_{*t}^{-1}(z))}{v_z(R(z), z, Z_{*t}^{-1}(z))} = (\partial_z\bar{R})(\bar{R}^{-1}(R(z), z, Z_{*t}^{-1}(z)), z, Z_{*t}^{-1}(z))$$

Arc-length trajectory

Let ϕ be such that

$$\phi(s) := \Phi_*(x, t(s)) \quad \text{and} \quad \phi(x,0) = \Phi_*(x,0) = x \quad \text{with} \quad \partial_s t(s) = |u|^{-1}.$$

Remark

$$|\partial_s \phi(s)| = 1$$

- $\tau(s)$: unit tangent vector
- $n(s)$: unit normal vector, $\kappa(s)$: curvature
- $b(s)$: unit binormal vector, $T(s)$: torsion

Key estimates on curvature and torsion

We can estimate κ, $\partial_s \kappa$, T by using Θ'' and Θ'''.

Up to here, we never touch the Euler equations with the pressure term. From next, we try to estimate the pressure term (totally separate from the above calculation).

In what follows, we use a differential geometric idea. See Chan-Czubak-Y (2014), more originally, see Ma-Wang (2004).

For any point $x \in \mathbb{R}^3$ near the arc-length trajectory ϕ is uniquely expressed as $x = \phi(\bar{\theta}) + \bar{r}n(\bar{\theta}) + \bar{z}b(\bar{\theta})$ with $(\bar{\theta}, \bar{r}, \bar{z}) \in \mathbb{R}^3$ (the meaning of the parameters s and $\bar{\theta}$ are the same along the arc-length trajectory).

Thus we have that

$$\begin{pmatrix} \partial_{\bar{\theta}} \\ \partial_{\bar{r}} \\ \partial_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 1-\kappa\bar{r} & \bar{z}\kappa & \bar{r}T \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ n \\ b \end{pmatrix}.$$

Therefore we have the following orthonormal moving frame: $\partial_{\bar{r}} = n$, $\partial_{\bar{z}} = b$ and

$$(1-\kappa\bar{r})^{-1}\partial_{\bar{\theta}} - \bar{z}T(1-\kappa\bar{r})^{-1}\partial_{\bar{r}} - \bar{r}T(1-\kappa\bar{r})^{-1}\partial_{\bar{z}} = \tau.$$

Rewrite the Euler equation along the particle trajectory

$$\nabla p \cdot \tau = \partial_\tau p = D_t|u| := \partial_t|u(\Phi_*(x,t),t)|$$

In general, pressure is nonlocal operator, nevertheless, we can extract the local pressure effect by using curvature and torsion.

Lemma (rewrite the pressure term using curvature and torsion)

Along the trajectory, we have (cf. Enciso and Peralta-Salas ARMA 2016, Kashiwabara-Notsu-Y in preparation)

$$3\kappa\partial_t|u| + \partial_s\kappa|u|^2 = \partial_{\bar r}\partial_t|u| \quad \text{and} \quad T\kappa|u|^2 = \partial_{\bar z}\partial_t|u|.$$

$$\text{Recall} \quad \partial_\tau = (1-\kappa\bar r)^{-1}\partial_{\bar\theta} - \bar z T(1-\kappa\bar r)^{-1}\partial_{\bar r} - \bar r T(1-\kappa\bar r)^{-1}\partial_{\bar z}.$$

The key is the pressure estimate:

$$\begin{aligned} -\partial_{\bar r}(\nabla p\cdot\tau) &= -\partial_{\bar r}\partial_\tau p = -\kappa\partial_{\bar\theta}p - \partial_{\bar r}\partial_{\bar\theta}p - T\partial_{\bar z}p \\ \text{(commute } \partial_{\bar r} \text{ and } \partial_{\bar\theta}) &= -\kappa(\nabla p\cdot\tau) - \partial_{\bar\theta}(\nabla p\cdot n) - T(\nabla p\cdot b) \end{aligned}$$

We can induce a contradiction from estimates of κ, $\partial_s\kappa$, T!!

Thank you!

Dziekuje!

Dekuju!

Xie Xie!

Komapsumnida!

Tesekkur ederim!

Merci beaucoup!

Danke!

Grazie!

Dekuji!

Tack!

Buiocas! Shukran!

Gracias! Chokrann!

Cam on!

Spasibo!