Nonlinear Science Seminar

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Lie Transform Perturbation Theory for Hamiltonian Systems and its Application to Guiding Center Motion

H. Sugama

National Institute for Fusion Science

SOKENDAI

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Hamiltonian Mechanics & Variational Principle

Action integral $$I = \int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\,dt$$

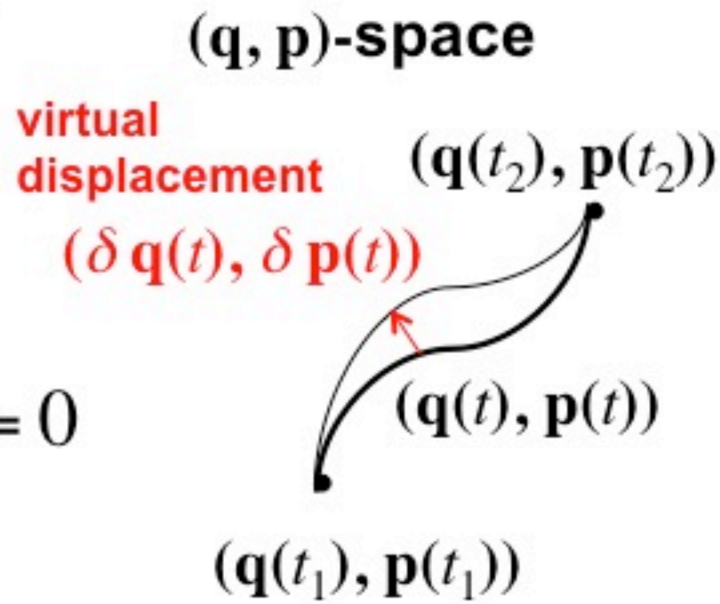
Variational principle for deriving Hamilton's equations :

The real path $(\mathbf{q}(t), \mathbf{p}(t))$ in phase space satisfies

$$\delta I = \delta \int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\,dt = 0$$

where end points are fixed :

$$\delta\mathbf{q}(t_1) = \delta\mathbf{p}(t_1) = \delta\mathbf{q}(t_2) = \delta\mathbf{p}(t_2) = 0$$



$$\delta I = \int_{t_1}^{t_2} \mathrm{d}t \left(\delta\,\mathbf{p}\cdot\dot{\mathbf{q}} + \mathbf{p}\cdot\delta\,\dot{\mathbf{q}} - \delta\,\mathbf{q}\cdot\frac{\partial H}{\partial\,\mathbf{q}} - \delta\,\mathbf{p}\cdot\frac{\partial H}{\partial\,\mathbf{p}} \right)$$

$$= \int_{t_1}^{t_2} \mathrm{d}t \left[-\delta\,\mathbf{q}\cdot\left(\dot{\mathbf{p}} + \frac{\partial H}{\partial\,\mathbf{q}} \right) + \delta\,\mathbf{p}\cdot\left(\dot{\mathbf{q}} - \frac{\partial H}{\partial\,\mathbf{p}} \right) \right]$$

$$= 0$$

$$\Downarrow$$

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{p}} \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{q}}$$

Hamiltonian Mechanics in Noncanonical Coordinates

Canonical coordinates $(\mathbf{q}, \mathbf{p})$ **in phase space**

$$\mathbf{q} = (q^i)_{i=1,\cdots,n}$$

$$\mathbf{p} = (p_i)_{i=1,\cdots,n}$$

$$L(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, t) = \mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)$$

$$I = \int_{t_1}^{t_2} L(\mathbf{q}(t), \mathbf{p}(t), \dot{\mathbf{q}}(t), t)\,\mathrm{d}t$$

General (noncanonical) coordinates $\mathbf{z} = (z^i)_{i=1,\cdots,2n}$

$$\mathbf{z} = \mathbf{z}(\mathbf{q}, \mathbf{p}, t) \quad \Longleftrightarrow \quad \mathbf{q} = \mathbf{q}(\mathbf{z}, t), \;\; \mathbf{p} = \mathbf{p}(\mathbf{z}, t)$$

$$L(\mathbf{z}, \dot{\mathbf{z}}, t) = \boldsymbol{\gamma}(\mathbf{z}, t) \cdot \dot{\mathbf{z}} - h(\mathbf{z}, t)$$

where $\boldsymbol{\gamma}(\mathbf{z}, t) = (\gamma_i(\mathbf{z}, t))_{i=1,\cdots,2n}$

$$\gamma_i(\mathbf{z}, t) = \mathbf{p}(\mathbf{z}, t) \cdot \frac{\partial \mathbf{q}(\mathbf{z}, t)}{\partial z^i},$$

$$h_{\mathrm{can}}(\mathbf{q}, \mathbf{p}, \mathrm{t}) = H(\mathbf{q}, \mathbf{p}, \mathrm{t})$$

$$h(\mathbf{z}, t) = h_{\mathrm{can}}(\mathbf{q}(\mathbf{z}, t), \mathbf{p}(\mathbf{z}, t), t) - \mathbf{p}(\mathbf{z}, t) \cdot \frac{\partial \mathbf{q}(\mathbf{z}, t)}{\partial t}$$

Variational principle in general (noncanonical) coordinates

$$\delta I = \delta \int_{t_1}^{t_2} L(\mathbf{z}(t), \dot{\mathbf{z}}(t), t)\, \mathrm{d}t = 0$$

$$L(\mathbf{z}, \dot{\mathbf{z}}, t) = \boldsymbol{\gamma}(\mathbf{z}, t) \cdot \dot{\mathbf{z}} - h(\mathbf{z}, t)$$

Euler-Lagrange eqs. $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{z}}}\right) - \frac{\partial L}{\partial \mathbf{z}} = 0$

⇩

$$\sum_{j=1}^{2n} \omega_{ij} \frac{\mathrm{d}z^j}{\mathrm{d}t} = \frac{\partial h}{\partial z^i} + \frac{\partial \gamma_i}{\partial t}$$

where $\omega_{ij} = \frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j} \qquad (i, j = 1, \cdots, 2n)$

$$\omega_{ij} = \frac{\partial \mathbf{p}}{\partial z^i} \cdot \frac{\partial \mathbf{q}}{\partial z^j} - \frac{\partial \mathbf{p}}{\partial z^j} \cdot \frac{\partial \mathbf{q}}{\partial z^i} = [z^i, z^j]_L$$

Lagrange brackets

Lagrange tensor $$\boxed{\omega_{ij} = \frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j}}$$

$$\omega_{ij} = -\omega_{ji} \qquad \frac{\partial \omega_{jk}}{\partial z^i} + \frac{\partial \omega_{ki}}{\partial z^j} + \frac{\partial \omega_{ij}}{\partial z^k} = 0$$

Poisson tensor $$\boxed{J^{ij} = \{z^i, z^j\} = \sum_{\alpha=1}^{n} \left(\frac{\partial z^i}{\partial q^\alpha} \frac{\partial z^j}{\partial p_\alpha} - \frac{\partial z^i}{\partial p_\alpha} \frac{\partial z^j}{\partial q^\alpha} \right)}$$

$$J^{ij} = -J^{ji} \qquad \sum_{k=1}^{2n} J^{ik} \omega_{kj} = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

Poisson brackets $$\{F, G\} = \sum_{i,j} J^{ij} \frac{\partial F}{\partial z^i} \frac{\partial G}{\partial z^j}$$

Jacobi identity $$\{F,\{G, H\}\} + \{G,\{H, F\}\} + \{H,\{F, G\}\} = 0$$

$$\{z^i, \{z^j, z^k\}\} + \{z^j, \{z^k, z^i\}\} + \{z^k, \{z^i, z^j\}\}$$

$$= \sum_l \left(J^{il} \frac{\partial J^{jk}}{\partial z^l} + J^{jl} \frac{\partial J^{ki}}{\partial z^l} + J^{kl} \frac{\partial J^{ij}}{\partial z^l} \right) = 0$$

Canonical coordinates $(\mathbf{q}, \mathbf{p})$ as a special case $(\mathbf{z})$

$$\gamma_{\mathbf{q}}^{(\mathrm{can})} = (\gamma_{q^\alpha}^{(\mathrm{can})})_{\alpha=1,\cdots,n} = \mathbf{p} \qquad \gamma_{\mathbf{p}}^{(\mathrm{can})} = (\gamma_{p_\alpha}^{(\mathrm{can})})_{\alpha=1,\cdots,n} = 0$$

Lagrange tensor

$$\omega_{q^\alpha p_\beta}^{(\mathrm{can})} = -\omega_{p_\alpha q^\beta}^{(\mathrm{can})} = -\delta_{\alpha\beta}$$

$$\omega_{q^\alpha q^\beta}^{(\mathrm{can})} = \omega_{p_\alpha p_\beta}^{(\mathrm{can})} = 0 \qquad (\alpha, \beta = 1, \cdots, n)$$

Poisson brackets

$$\{q^\alpha, q^\beta\} = \{p_\alpha, p_\beta\} = 0,$$

$$\{q^\alpha, p_\beta\} = -\{p_\alpha, q^\beta\} = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1, \cdots, n)$$

Equations of motion in general (noncanonical) coordinates ($\mathbf{z}$)

$$\sum_{j=1}^{2n} \omega_{ij} \frac{\mathrm{d}z^j}{\mathrm{d}t} = \frac{\partial h}{\partial z^i} + \frac{\partial \gamma_i}{\partial t}$$

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \sum_j J^{ij} \left(\frac{\partial h}{\partial z^j} + \frac{\partial \gamma_j}{\partial t} \right) = \{ z^i, h \} + \sum_j \{ z^i, z^j \} \frac{\partial \gamma_j}{\partial t}$$

phase space volume element $\mathrm{d}q^1 \cdots \mathrm{d}q^n \, \mathrm{d}p_1 \cdots \mathrm{d}p_n = D \, \mathrm{d}z^1 \cdots \mathrm{d}z^{2n}$

Jacobian $D = \det \left[\frac{\partial (q^1, \cdots, q^n, p_1, \cdots, p_n)}{\partial (z^1, \cdots, z^{2n})} \right]$ $\qquad \det(\omega_{ij}) = D^2$

Liouville's theorem :

conservation of phase space volume element

$$\frac{\partial D}{\partial t} + \sum_{i=1}^{2n} \frac{\partial (D \dot{z}^i)}{\partial z^i} = 0$$

Motion of a Charged Particle in Electromagnetic Fields

Electromagnetic fields $\mathbf{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$

Hamiltonian in canonical coordiantes $(\mathbf{q}, \mathbf{p})$

$$h_{\mathrm{can}}(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2m}\left|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q}, t)\right|^2 + e\Phi(\mathbf{q}, t)$$

Canonical coordinates $(\mathbf{q}, \mathbf{p})$ $\Longleftrightarrow$ **Noncanonical coordinates** $\mathbf{z} = (\mathbf{x}, \mathbf{v})$

$$\mathbf{q} = \mathbf{x}, \qquad \mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t)$$

Lagrangian and Hamiltonian in noncanonical coordinates
$\mathbf{z} = (\mathbf{x}, \mathbf{v})$

$$L(\mathbf{z}, \dot{\mathbf{z}}, t) = \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right) \cdot \dot{\mathbf{x}} - h(\mathbf{z}, t)$$

$$h(\mathbf{z}, t) = \frac{1}{2}m|\mathbf{v}|^2 + e\Phi(\mathbf{x}, t)$$

$$L(\mathbf{z},\dot{\mathbf{z}},t)=\boldsymbol{\gamma}(\mathbf{z},t)\cdot\dot{\mathbf{z}}-h(\mathbf{z},t)=\left(m\,\mathbf{v}+\frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)\cdot\dot{\mathbf{x}}-h(\mathbf{z},t)$$

$$\gamma_{\mathbf{x}}=m\,\mathbf{v}+\frac{e}{c}\mathbf{A}(\mathbf{x},t),\qquad \boldsymbol{\gamma}_{\mathbf{v}}=0$$

Lagrange tensor $\omega_{ij}=\dfrac{\partial\gamma_j}{\partial z^i}-\dfrac{\partial\gamma_i}{\partial z^j}$

$$\omega_{x_\alpha x_\beta}=\frac{e}{c}\left(\frac{\partial A_\beta}{\partial x_\alpha}-\frac{\partial A_\alpha}{\partial x_\beta}\right)=\frac{e}{c}\sum_{\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}B_\gamma,\quad \varepsilon_{\alpha\beta\gamma}=\begin{cases}1\,((\alpha,\beta,\gamma)=(1,2,3),(2,3,1),(3,1,2))\\-1\,((\alpha,\beta,\gamma)=(1,3,2),(2,1,3),(3,2,1))\\0\,(\text{otherwise})\end{cases}$$

$$\omega_{x_\alpha v_\beta}=-m\delta_{\alpha\beta},\qquad \omega_{v_\alpha v_\beta}=0\qquad(\alpha,\beta=1,2,3)$$

Poisson tensor $J^{ij}=\{z^i,z^j\}=\displaystyle\sum_{\alpha=1}^{n}\left(\frac{\partial z^i}{\partial q^\alpha}\frac{\partial z^j}{\partial p_\alpha}-\frac{\partial z^i}{\partial p_\alpha}\frac{\partial z^j}{\partial q^\alpha}\right)$

Poisson brackets $\{x_\alpha,x_\beta\}=0,\qquad \{x_\alpha,v_\beta\}=\dfrac{1}{m}\delta_{\alpha\beta},$

$$\{v_\alpha,v_\beta\}=\frac{e}{m^2c}\sum_{\gamma=1}^{3}\varepsilon_{\alpha\beta\gamma}B_\gamma\qquad(\alpha,\beta=1,2,3)$$

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \sum_j J^{ij} \left(\frac{\partial h}{\partial z^j} + \frac{\partial \gamma_j}{\partial t} \right) = \{z^i, h\} + \sum_j \{z^i, z^j\} \frac{\partial \gamma_j}{\partial t}$$

$$\Downarrow$$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v},$$

$$m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e \left[\mathbf{E}(\mathbf{x}, t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{x}, t) \right]$$

Formulation of Hamiltonian Mechanics based on 1-Form

Fundamental 1-form on the $(2n+1)$-dimensional $(\mathbf{z}, t)$ space

$$\gamma = \sum_{\mu=1}^{2n+1} \gamma_\mu(z)\,\mathrm{d}z^\mu = \sum_{i=1}^{2n} \gamma_i(\mathbf{z}, t)\,\mathrm{d}z^i - h(\mathbf{z}, t)\,\mathrm{d}t$$

Action integral along the path l connecting $(\mathbf{z}_1, t_1)$ and $(\mathbf{z}_2, t_2)$

$$I = \int_l \gamma$$

The real path $\mathbf{z}(t)$ in phase space satisfies $\delta I = 0$

where end points are fixed : $\delta \mathbf{z}(t_1) = \delta \mathbf{z}(t_2) = 0$

The action integral I and the path $\mathbf{z}(t)$ derived from $\delta I = 0$
are invariant under the gauge transform $\gamma' = \gamma + \mathrm{d}S$

Where $S(\mathrm{z}, t)$ **is an arbitrary function.**

2-form ω derived from the exterior derivative of γ

$$\omega = \mathrm{d}\gamma = \sum_{i=1}^{2n} \mathrm{d}\gamma_i \wedge \mathrm{d}z^i - \mathrm{d}h \wedge \mathrm{d}t$$
$$= \sum_{i<j} \omega_{ij} \mathrm{d}z^i \wedge \mathrm{d}z^j - \sum_i \left(\frac{\partial \gamma_i}{\partial t} + \frac{\partial h}{\partial z^i} \right) \mathrm{d}z^i \wedge \mathrm{d}t$$

Put $\mathrm{d}t = 0$ in ω to define Lagrange (covariant) tensor on z-space

$$\hat{\omega} = \sum_{i<j} \omega_{ij} \mathrm{d}z^i \wedge \mathrm{d}z^j$$

Map φ : z-space to itself is symplectic (or canonical) iff

$$\varphi^* \hat{\omega} = \hat{\omega}$$

Poisson (contravariant) tensor on z-space is defined by

$$J = \sum_{i,j} J^{ij} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^i} = \sum_{i<j} J^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^i}$$

where (J^{ij}) is the inverse matrix of (ω_{ij}).

Transformation of Space, Coordinates & Functions

m**-dimensional space** M **(manifold)**

Transformation $T : M \ni a \longrightarrow T(a) \in M$

Two coordinate systems are related each other by

$$z : M \ni a \to z(a) = (z^{\mu}(a))_{i=1,\cdots,m} \in \mathbb{R}^m$$

$$\bar{z} : M \ni a \to \bar{z}(a) = (\bar{z}^{\mu}(a))_{i=1,\cdots,m} \in \mathbb{R}^m$$

$$\bar{z} = T^* z \equiv z \circ T$$

$$\bar{z}(a) = (T^* z)(a) \equiv z(T(a))$$

$$\bar{z}^{\mu}(a) = (T^* z^{\mu})(a) \equiv z^{\mu}(T(a)) \quad (i = 1, \cdots, m)$$

Function on M $f : M \ni a \to f(a) \in \mathbb{R}$

is represented by

$F : \mathbb{R}^m \ni w = (w^{\mu})_{\mu=1,\cdots,m} \to F(w) \in \mathbb{R}$ **in** z **-coordinates**

$\overline{F} : \mathbb{R}^m \ni w = (w^{\mu})_{\mu=1,\cdots,m} \to \overline{F}(w) \in \mathbb{R}$ **in** $\overline{z}$**-coordinates**

$$f(a) = F(z(a)) = \overline{F}(\bar{z}(a))$$

$$f(a) = F(z(a)) = \boxed{\bar{F}(\bar{z}(a))}$$

$$((T^{-1})^* f)(a) \equiv f(T^{-1}(a)) = \boxed{\bar{F}(z(a))}$$

Two pictures (or interpretations) for the function $\bar{F}$

Passive picture :

$\bar{F}$ **represents** f **in** $\bar{z}$ **-coordinates and transfomation is done on coordinates.**

Active picture :

$\bar{F}$ **represents** $(T^{-1})^* f$ **in** z **-coordinates and transfomation is done on a function.**

1-form $$\gamma = \sum_{\mu} \gamma_{\mu}(z)\, \mathrm{d}z^{\mu} = \sum_{\nu} \overline{\gamma}_{\nu}(\overline{z})\, \mathrm{d}\overline{z}^{\nu}$$

Two representations of 1-form γ in two coordinate systems

$$\gamma_{\mu}(z(a)) = \gamma_a\left(\left(\frac{\partial}{\partial z^{\mu}}\right)_a\right) \qquad \boxed{\overline{\gamma}_{\nu}(\overline{z}(a)) = \gamma_a\left(\left(\frac{\partial}{\partial \overline{z}^{\nu}}\right)_a\right)}$$

Passive picture :

$\overline{\gamma}_{\mu}$ represents γ in $\overline{z}$ -coordinates and transfomation is done on coordinates.

$$\boxed{\overline{\gamma}_{\nu}(z(a)) = ((T^{-1})^{*}\gamma)_a\left(\left(\frac{\partial}{\partial z^{\nu}}\right)_a\right)}$$

Active picture :

$\overline{\gamma}_{\mu}$ represents $(T^{-1})^{*}\gamma$ in z-coordinates and transfomation is done on 1-form.

Lie Transform Perturbation Theory

T **: Transformation of (2*n*+1)-dimensional space to itself**

Transformation of coordnates $\bar{z}^{\mu} = T^{*} z^{\mu} = z^{\mu} \circ T \quad (\mu = 1, \cdots, 2n+1)$

Transformation of fundamental 1-form $\bar{\gamma} = (T^{-1})^{*} \gamma + dS = \sum_{\mu=1}^{2n+1} \bar{\gamma}_{\mu}(z)\, dz^{\mu}$

We want T **such that** $\bar{\gamma}_{\mu}(z)$ **takes a convenient form.**

For example, if $(\bar{\gamma}_{\mu})_{\mu=1,\cdots,2n+1}$ **is independent of** z^{λ}**,**
then $\bar{\gamma}^{\lambda}$ **becomes an invariant of motion.**

We use T **which takes the form** $T = \cdots T_3 T_2 T_1$

where $T_n = \mathrm{Exp}(\varepsilon^{n} G_n)$ **: Lie transformation**

G_n **: vector field (differential operator)**

Lie transformation $T_n = \mathrm{Exp}(\varepsilon^n G_n)$ G_n **: a vector field**

Pushforward (differential) of T_n

$$(T_n)_* = \exp(-\varepsilon^n L_n) = 1 - \varepsilon L_n + \frac{\varepsilon^2}{2}(L_n)^2 + \cdots$$

Lie derivative $L_n = L_{Gn}$ **acting on a vector field** X

$$L_n X = [G_n, X] = G_n X - X G_n$$

Pullback of T_n

$$(T_n)^* = \exp(\varepsilon^n L_n) = 1 + \varepsilon L_n + \frac{\varepsilon^2}{2}(L_n)^2 + \cdots$$

Lie derivative $L_n = L_{Gn}$ **acting on a differential form** θ

$$L_n \theta = i(G_n)\,\mathrm{d}\theta + \mathrm{d}i(G_n)\,\theta$$

Map $T = \cdots T_3 T_2 T_1$ **yields**

Transformation of coordinates

$$\bar{z} = T^* z = (T_1)^* (T_2)^* (T_3)^* \cdots z$$
$$= \bar{z}_0 + \varepsilon \bar{z}_1 + \varepsilon^2 \bar{z}_2 + \cdots$$

$\Rightarrow$

$$\bar{z}_0 = z$$
$$\bar{z}_1 = G_1 z$$
$$\bar{z}_2 = \left(\frac{1}{2} (G_1)^2 + G_2 \right) z$$

Transformation of fundamental 1-form

$$\bar{\gamma} = (T^{-1})^* \gamma + dS$$

$\Rightarrow$

$$\bar{\gamma}_0 = \gamma_0 + \mathrm{d}S_0$$
$$\bar{\gamma}_1 = \gamma_1 - L_1 \gamma_0 + \mathrm{d}S_1$$
$$\bar{\gamma}_2 = \gamma_2 - L_1 \gamma_1 + \left(\frac{1}{2} (L_1)^2 - L_2 \right) \gamma_0 + \mathrm{d}S_2$$

We can use $L_n \gamma = i(G_n) d\gamma$ **by including** $d\, i(G_n)\, \gamma$ **into** dS

Determine G_n **and** S_n $(n=1,2, \ldots)$ **which make** $\bar{\gamma}_n$ **take desirable forms.**

Derivation of Guiding Center Motion Equations

Fundamental 1-form for a charged particle in the electromagnetic field

$$\gamma = \left(\varepsilon^{-1} \frac{e}{c} \mathbf{A}(\mathbf{x}, \varepsilon t) + m\,\mathbf{v} \right) \cdot \mathrm{d}\mathbf{x} - h\,\mathrm{d}t$$

Hamiltonian $$h = \frac{1}{2} m |\mathbf{v}|^2 + e\Phi(\mathbf{x}, \varepsilon t)$$

$$\varepsilon << 1$$

$$e\Phi \sim mv^2$$

Motion equations

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}, \\ m\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\,\mathbf{E}(\mathbf{x}, \varepsilon t) + \varepsilon^{-1}\dfrac{e}{c}\mathbf{v} \times \mathbf{B}(\mathbf{x}, \varepsilon t) \end{cases}$$

$$e\,\mathbf{E}(\mathbf{x}, \varepsilon t) = -e\,\nabla\Phi(\mathbf{x}, \varepsilon t) - \frac{e}{c}\frac{\partial \mathbf{A}}{\partial \tau}(\mathbf{x}, \varepsilon t) \qquad \tau = \varepsilon t$$

Drift ordering parameter $$\varepsilon \sim \frac{cE}{vB} \sim \frac{c\Phi}{vBl} \sim \frac{mcv}{eBl} \sim \frac{\rho}{l}$$

Coordinates $z=(z^{\mu})_{\mu=1,\cdots,7}=(\mathbf{x}, v_{\parallel}, \theta, v_{\perp}, \tau)$

particle position $\mathbf{x}$

θ**: gyrophase**

particle velocity

$$\mathbf{v}=v_{\parallel}\mathbf{b}(\mathbf{x},\tau)+v_{\perp}\mathbf{c}(\mathbf{x},\tau,\theta)$$
$$=v_{\parallel}\mathbf{b}(\mathbf{x},\tau)-v_{\perp}[\sin\theta\,\mathbf{e}_1(\mathbf{x},\tau)+\cos\theta\,\mathbf{e}_2(\mathbf{x},\tau)]$$

Fundamental 1-form in the z coordinates

$$\gamma=\sum_{\mu}\gamma_{\mu}(z)\,\mathrm{d}z^{\mu}=\varepsilon^{-1}\gamma^{(0)}+\gamma^{(1)}$$

$$\gamma^{(0)}=\frac{e}{c}\mathbf{A}(\mathbf{x},\tau)\cdot\mathrm{d}\mathbf{x}-\left(\frac{1}{2}m(v_{\parallel}^2+v_{\perp}^2)+e\Phi(\mathbf{x},\tau)\right)\mathrm{d}\tau$$

$$\gamma^{(1)}=m\left[v_{\parallel}\mathbf{b}(\mathbf{x},\tau)+v_{\perp}\mathbf{c}(\mathbf{x},\tau)\right]\cdot\mathrm{d}\mathbf{x}$$

Use Lie transform perturbation theory to derive new coordinates in which γ is independent of the gyrophase dependence.

Then, the magnetic moment, which is the momentum conjugate to the gyrophase, becomes an invariant.

Preparatory Lie transform $T^{(p)} = \mathrm{Exp}(\varepsilon G^{(p)})$

$$G^{(\mathrm{p})} = -\frac{cmv_\perp}{eB}\mathbf{a}\cdot\left(\nabla + \mathrm{R}\frac{\partial}{\partial\theta}\right) \qquad \mathbf{a} = \mathbf{b}\times\mathbf{c} \qquad \mathbf{R} = \nabla\mathbf{e}_1\cdot\mathbf{e}_2$$

$$\mathbf{x}^{(\mathrm{p})} = (T^{(\mathrm{p})})^*\mathbf{x} = \mathbf{x} - \varepsilon\frac{cmv_\perp}{eB(\mathbf{x},\tau)}\mathbf{a}(\mathbf{x},\tau) + \mathcal{O}(\varepsilon^2),$$

$$\theta^{(\mathrm{p})} = (T^{(\mathrm{p})})^*\theta = \theta + \varepsilon(G^{(\mathrm{p})})^{\mathbf{x}}\cdot\mathbf{R}(\mathbf{x},\tau) + \mathcal{O}(\varepsilon^2),$$

$$v_\parallel^{(\mathrm{p})} = (T^{(\mathrm{p})})^* v_\parallel = v_\parallel,$$

$$v_\perp^{(\mathrm{p})} = (T^{(\mathrm{p})})^* v_\perp = v_\perp,$$

$$\tau^{(\mathrm{p})} = (T^{(\mathrm{p})})^*\tau = \tau$$

$-\frac{cmv_\perp}{eB}\mathbf{a}$: **Gyroradius of lowest order in** ε

$\mathbf{R} = \nabla\mathbf{e}_1\cdot\mathbf{e}_2$ **depends on how to choose** $(\mathbf{e}_1, \mathbf{e}_2)$ **[gyrogauge].**

Preparatory Lie transform of fundamental 1-form

$$\bar{\gamma} = ((T^{(p)})^{-1})^{*}\gamma = \mathrm{Exp}(-\varepsilon L_p)\,\gamma = \varepsilon^{-1}\sum_{n=0}^{\infty}\varepsilon^{n}\bar{\gamma}^{(n)} \qquad L_p = i(G^{(p)})d$$

0th order

$$\bar{\gamma}^{(0)} = \gamma^{(0)} = \frac{e}{c}\mathbf{A}(\mathbf{x},\tau)\cdot \mathrm{d}\mathbf{x} - \left(\frac{1}{2}m(v_{\parallel}^{2}+v_{\perp}^{2}) + e\Phi(\mathbf{x},\tau)\right)\mathrm{d}\tau$$

1st order

$$\bar{\gamma}^{(1)} = \gamma^{(1)} - L_{\mathrm{p}}\gamma^{(0)} = mv_{\parallel}\mathbf{b}\cdot\mathrm{d}\mathbf{x} + \frac{cmv_{\perp}}{B}\mathbf{a}\cdot\mathbf{E}\mathrm{d}\tau$$

2nd order

$$\bar{\gamma}^{(2)} = -L_{\mathrm{p}}\gamma^{(1)} + \frac{1}{2}L_{\mathrm{p}}^{2}\gamma^{(0)} = \sum_{\mu}\bar{\gamma}_{\mu}^{(2)}\mathrm{d}z^{\mu}$$

$$\bar{\gamma}_{\mathbf{x}}^{(2)} = -\frac{m^{2}cv_{\perp}^{2}}{2eB}[\mathbf{R} + (\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c})\mathbf{b}] - \frac{m^{2}cv_{\parallel}v_{\perp}}{eB}[(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a})\mathbf{b} + (\mathbf{b}\cdot\nabla\times\mathbf{b})\mathbf{c}],$$

$$\bar{\gamma}_{\theta}^{(2)} = \frac{m^{2}cv_{\perp}^{2}}{2eB}, \qquad \bar{\gamma}_{v_{\parallel}}^{(2)} = \bar{\gamma}_{v_{\perp}}^{(2)} = 0$$

Magnetic moment of lowest order

depends on gyrophase θ

Further Lie transform

$$T = \cdots T_3 T_2 T_1 \qquad\qquad T_n = \mathrm{Exp}(\varepsilon^n G_n)$$

Transformed fundamental 1-form

$$\Gamma = (T^{-1})^* \overline{\gamma} + dS = \varepsilon^{-1} \sum_{n=0}^{\infty} \varepsilon^n \Gamma^{(n)}$$

Determine G_n and S_n $(n=1,2,\ldots)$ which make $\Gamma^{(n)}$ independent of gyrophase θ.

The new coordinates obtained from the Lie transform satisfy the guiding center motion equations in which the equation for the gyrophase is decoupled and the magnetic moment (the variable conjugate to the gyro phase) is conserved.

The results are shown in the following pages.

Guiding center coordinates $(\mathbf{X}, U, \Theta, \mu, \tau)$

$$\mathbf{X} = \mathbf{x} - \varepsilon \frac{mcv_\perp}{eB}\mathbf{a}$$

$$+\varepsilon^2 \frac{c^2 m^2}{e^2}\left(\left[-\frac{v_\perp^2}{2B^3}(\mathbf{a}\cdot\nabla B) + \frac{v_\parallel v_\perp}{B^2}(\mathbf{b}\cdot\nabla\times\mathbf{b})\right]\mathbf{a}\right.$$

$$\left.+\left[-\frac{v_\perp^2}{8B^2}(5\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) + \frac{2v_\parallel v_\perp}{B^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c})\right]\mathbf{b}\right)$$

$$+\mathcal{O}(\varepsilon^3)$$

$$U = v_\parallel + \varepsilon G_1^{v_\parallel} + \mathcal{O}(\varepsilon^2)$$

$$G_1^{v_\parallel} = -\frac{cmv_\perp^2}{2eB}(\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) - \frac{cmv_\parallel v_\perp}{eB}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a})$$

$$\Theta = \theta + \varepsilon\left[-\frac{cmv_\perp}{eB}(\mathbf{a}\cdot\mathbf{R}) + G_1^\theta\right] + \mathcal{O}(\varepsilon^2)$$

$$G_1^\theta = \frac{cmv_\perp}{eB^2}(\mathbf{c}\cdot\nabla B) - \frac{c}{v_\perp B}(\mathbf{c}\cdot\mathbf{E})$$

$$-\frac{cmv_\parallel}{4eB}(\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) + \frac{cmv_\parallel^2}{ev_\perp B}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c})$$

$$\mu = \frac{mv_\perp^2}{2B(\mathbf{x},\tau)} + \varepsilon\frac{cm^2}{e}\left[-\frac{ev_\perp}{mB^2}(\mathbf{a}\cdot\mathbf{E})\right.$$

$$+\frac{v_\parallel v_\perp^2}{4B^2}(3\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{a})$$

$$\left.+\frac{v_\parallel^2 v_\perp}{B^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) + \frac{v_\perp^3}{2B^3}(\mathbf{a}\cdot\nabla B)\right] + \mathcal{O}(\varepsilon^2)$$

Guiding center coordinates $(\mathbf{X}, U, \Theta, \mu, \tau)$

Fundamental 1-form

$$\gamma = \varepsilon^{-1}\frac{e}{c}\mathbf{A}^*(\mathbf{X}, U, \mu, \tau)\cdot \mathrm{d}\mathbf{X} + \varepsilon\frac{mc}{e}\mu\,\mathrm{d}\theta - \varepsilon^{-1}H(\mathbf{X}, U, \mu, \tau)\,\mathrm{d}\tau$$

where

$$\mathbf{A}^*(\mathbf{X}, U, \mu, \tau) = \mathbf{A}(\mathbf{X}, \tau) + \varepsilon\frac{mc}{e}U\,\mathbf{b}(\mathbf{X}, \tau) - \varepsilon^2\frac{mc^2}{e^2}\mu\,\mathbf{R}(\mathbf{X}, \tau)$$

Hamiltonian

$$H(\mathbf{X}, U, \mu, \tau) = \frac{1}{2}mU^2 + \mu B(\mathbf{X}, \tau) + e\Phi(\mathbf{X}, \tau) + \varepsilon\frac{mc\mu U}{2e}(\mathbf{b}\cdot\nabla\times\mathbf{b}) + O(\varepsilon^2)$$

Lagrangian

$$L\left(\mathbf{X}, U, \mu, \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau}, \frac{\mathrm{d}\Theta}{\mathrm{d}\tau}, \varepsilon\tau\right) = \varepsilon^{-1}\frac{e}{c}\mathbf{A}^*(\mathbf{X}, U, \mu, \tau)\cdot\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} + \varepsilon\frac{mc}{e}\mu\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} - \varepsilon^{-1}H(\mathbf{X}, U, \mu, \tau)$$

Poisson brackets

$$\left\{X_\alpha, X_\beta\right\} = -\varepsilon \frac{c}{eB_\parallel^*} \sum_{\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} b_\gamma \qquad \{\mathbf{X}, U\} = -\{U, \mathbf{X}\} = \frac{\mathbf{B}^*}{mB_\parallel^*}$$

$$\{\mathbf{X}, \Theta\} = -\{\Theta, \mathbf{X}\} = \varepsilon \frac{c}{eB_\parallel^*} \mathbf{b} \times \mathbf{R} \qquad \{U, \Theta\} = -\{\Theta, U\} = -\frac{\mathbf{B}^* \cdot \mathbf{R}}{mB_\parallel^*}$$

$$\{\Theta, \mu\} = -\{\mu, \Theta\} = \varepsilon^{-1} \frac{e}{mc}$$

where

$$\mathbf{B}^*(\mathbf{X}, U, \mu, \tau) = \nabla \times \mathbf{A}^*(\mathbf{X}, U, \mu, \tau)$$

$$B_\parallel^*(\mathbf{X}, U, \mu, \tau) = \mathbf{B}^*(\mathbf{X}, U, \mu, \tau) \cdot \mathbf{b}(\mathbf{X}, \tau)$$

Equations of guiding center motion

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{1}{B_\parallel^*}\left[U^*\mathbf{B}^* + \varepsilon c\,\mathbf{b}\times\left(\frac{\mu}{e}\nabla B - \mathbf{E}^*\right)\right]$$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\mathbf{B}^*}{mB_\parallel^*}\cdot(\mu\nabla B - e\,\mathbf{E}^*)$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \varepsilon^{-1}\frac{eB}{mc} + \frac{U}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}) + \mathbf{R}\cdot\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

No dependence on the gyrophase Θ

Magnetic moment

$\mu = \text{const}$

where

$$U^* = U + \varepsilon\frac{c}{2e}\mu(\mathbf{b}\cdot\nabla\times\mathbf{b})$$

$$\mathbf{E}^* = -\nabla\Phi^* - \frac{1}{c}\frac{\partial\mathbf{A}^*}{\partial\tau}$$

$$\Phi^* = \Phi + \varepsilon\frac{mc}{2e^2}\mu U(\mathbf{b}\cdot\nabla\times\mathbf{b})$$

The number of variables to be solved is reduced from six to four *!!*

Summary

The Lie transform perturbation theory is a convenient method to find noncanonical coordinates in which the fundamental 1-form and Hamiltonian's canonical equations are transformed into simpler structures.

As an example, derivation of the guiding center motion equations is shown.

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