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Local flow geometry of a vortex and associated physical quantities

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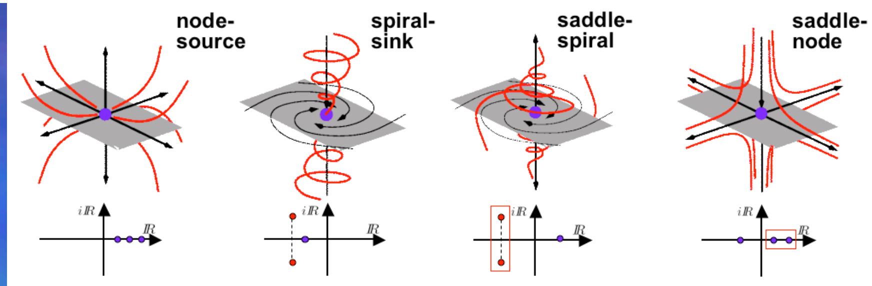
1. Introduction
1.1 Vortices in flow

(https://www.nrel.gov)

1. Vortices play important roles in turbulent flows and influences performance or soundness of facility, system or machinery in various engineering field, such as power plants, wind turbines, or aviation.
2. Identification of a vortex with its intensity is important, and clarifying the feature and mechanism of the vortex is necessary.
3. No universal definition has been established, and existence of vortices depends on an applied definition.



1.2 On vortex definition

(Garth et al., 2005)

1. Stream line is not appropriate because it is not Galilei invariant (depending on the inertial coordinate system).
2. Vorticity is difficult to distinguish the geometries (topologies) between shear and vortical (swirling) flows.
3. In many vortex definitions proposed, popular definitions frequently used are on local approach specified by the velocity gradient tensor $\nabla \boldsymbol{v}$ such as the local flow geometry (topology) or pressure minimum feature*.
4. the complex eigenvalues of $\nabla \boldsymbol{v}$ that specify the invariant vortical flow have greatly contributed in the vortex definitions (e.g., Δ definition) and classification of flows.

(*: Chong et al., *Phy. Fluids*, 1990, Hunt, STR-88, 1988, Jeong et al., *J. Fluid. Mech.*, 1995)

1.3 Questions on topology of a vortex

It has been desired to clarify the following questions:

1. Behind frequent application of the eigenvalues of $\nabla \boldsymbol{v}$, what is the clear interpretation of the eigenvalues of $\nabla \boldsymbol{v}$? (Is the classification correct?)
2. No identification of flow symmetry* ?
3. How are the pressure minimum in the vortex definitions and vortex stretching related to the local flow topology?
4. What is the universal definition of a vortex or vortical axis?

These items are now being clarified with a new aspect…

(*: Blackburn et al., *J. Fluid. Mech.,* 1996)

2. Local flow topology and quantities
2.1 flow specified by $\nabla \boldsymbol{v}$

The local flow around a point can be expressed as:

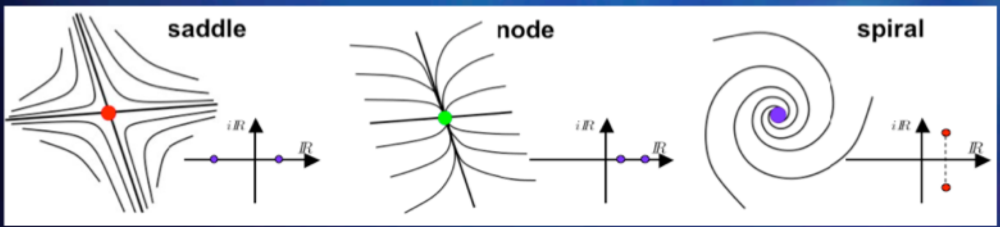
- Taylor expansion of velocity neglecting higher order:

$$\frac{d}{dt} x_i = \frac{\partial v_i}{\partial x_j} x_j$$ (summation convention is applied)

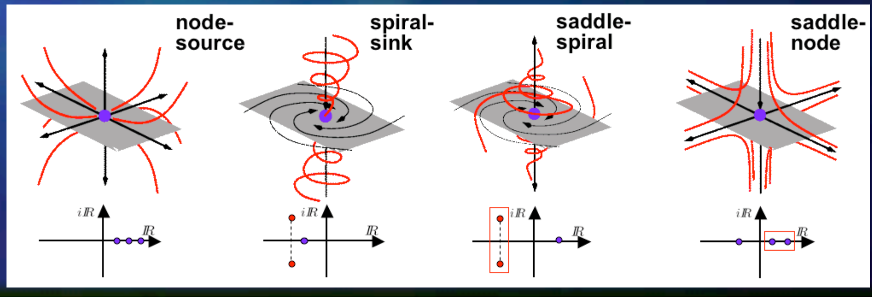
- Eigenvalues ε_i and eigenvectors ξ_i (i=1,2,3) of $\nabla \boldsymbol{v}$ specify the local flow geometry in terms of the Galilei invariant.

$$\boldsymbol{x} = \sum_{j=1}^{3} e^{\varepsilon_j t} \xi_j$$

a linear combination of flows along ξ_i (i=1,2,3), in the directions and with intensities according to ε_i .



(Garth et al., 2005)



2.2 Invariant vortical flow geometry

If $\nabla \boldsymbol{v}$ has complex conjugate and real eigenvalues, $\varepsilon_R \pm i\ \psi$ and ε_a, and their respective eigenvectors $\boldsymbol{\xi}_{pl} \pm i\ \boldsymbol{\eta}_{pl}$ and $\boldsymbol{\zeta}$,

➔ Flow trajectory*: $\boldsymbol{x} = 2\exp(\varepsilon_R t)\{\cos(\psi t)\boldsymbol{\xi}_{pl} - \sin(\psi t)\boldsymbol{\eta}_{pl}\} + \exp(\varepsilon_a t)\boldsymbol{\zeta}$

➔ Vortical flow

Flow proceeds (converges) along $\boldsymbol{\zeta}$.

Flow swirls in $\boldsymbol{\xi}_{pl}$-$\boldsymbol{\eta}_{pl}$ plane (swirl plane $\mathcal{P}$).

Inflow (converge) or outflow (diverge) according to ε_R .

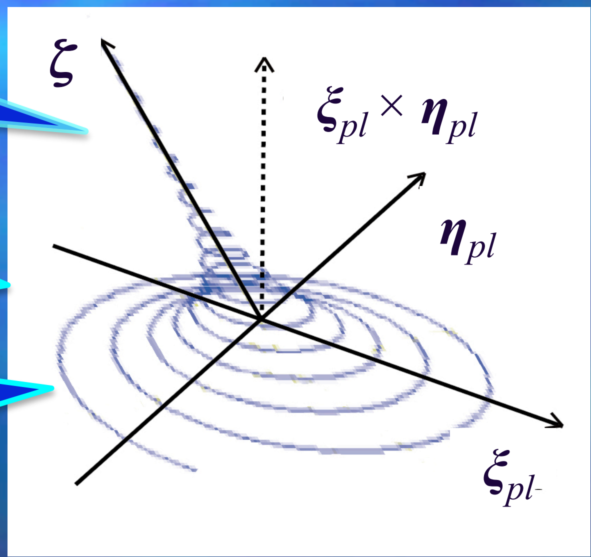
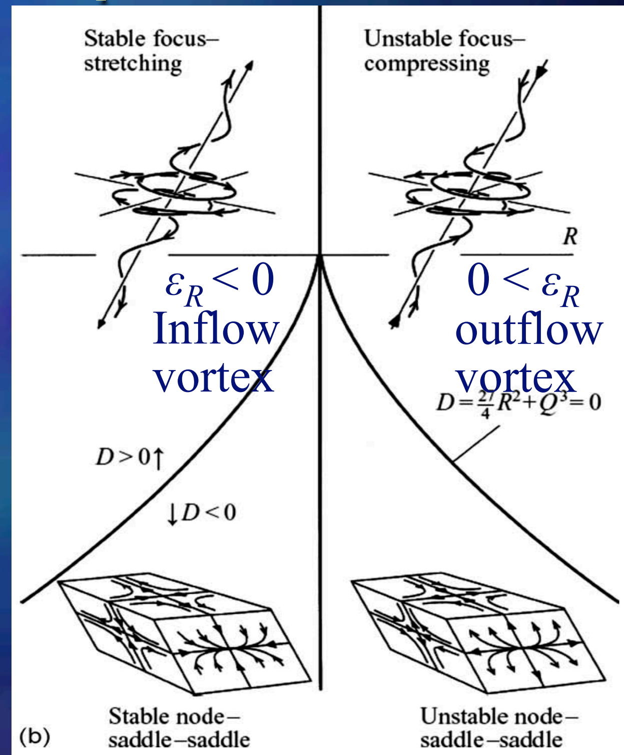


Fig. 2.1: vortical flow topology ($\varepsilon_R < 0$)



(*: Nakayama, *Fluid Dyn. Res.*, 2014)

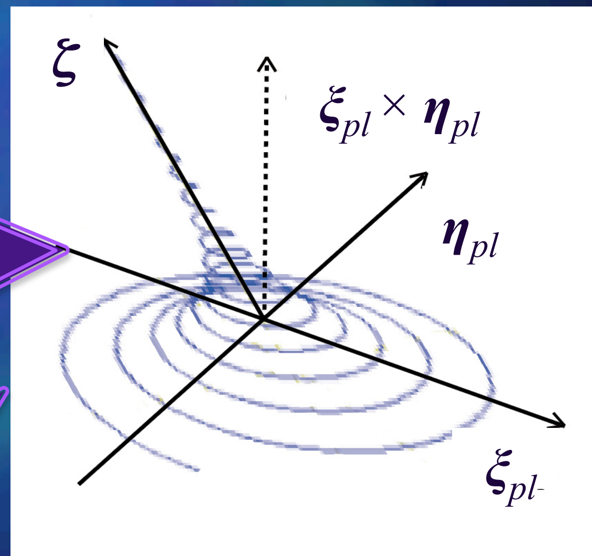
(Wallace, *Phy. Fluids* 2009)

2.3 Questions may arise…

- This flow classification has been applied in several turbulent flows and in the major vortex definitions since around 1990 .
- However….. $\boldsymbol{x} = 2\exp(\varepsilon_R t)\{\cos(\psi t)\boldsymbol{\xi}_{pl} - \sin(\psi t)\boldsymbol{\eta}_{pl}\} + \exp(\varepsilon_a t)\boldsymbol{\zeta}$

Does it swirl uniformly with constant ψ around a point?
Does it converge (inflow) or diverge (outflow) uniformly around a point?

What is the physical interpretation of the eigenvalues?
How is the flow symmetry?



(Fig. 2.1: vortical flow topology ($\varepsilon_R < 0$))

3. Exploring the local flow in a plane

- We study further detail of the topology in a plane.
- local velocity:

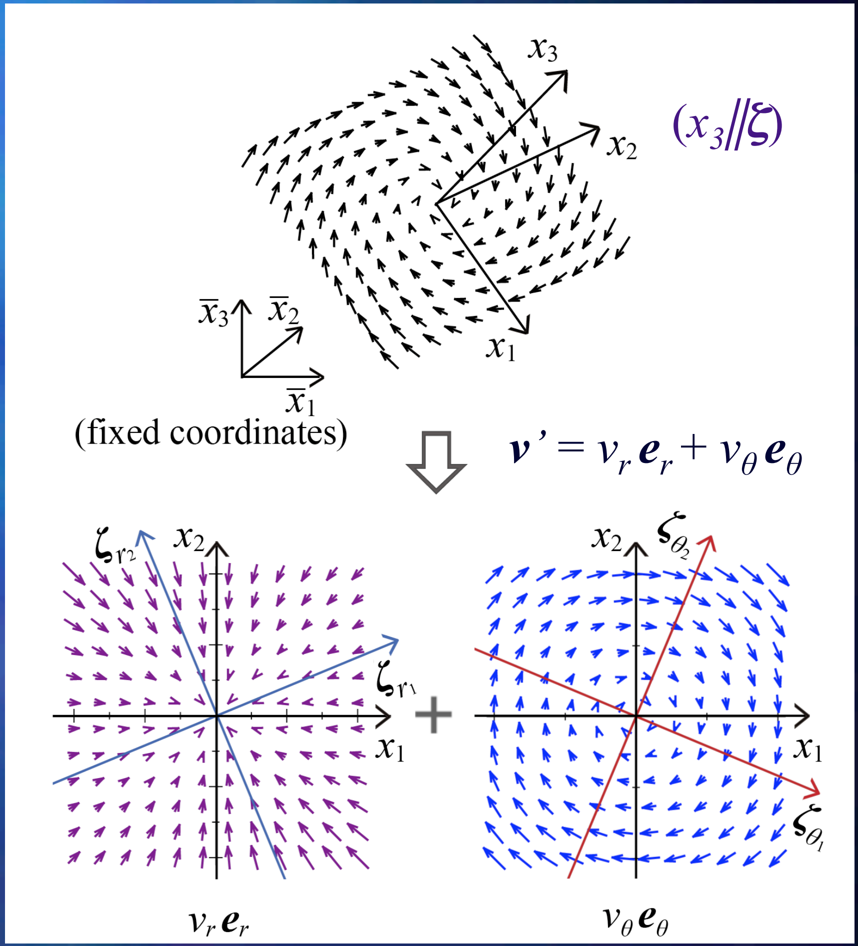
  $v'_i = (\partial v_i / \partial x_j)\, x_j \; (i, j = 1, 2)$

➢ <u>decompose the flow</u> into
  (i) azimuthal flow v_θ
  (ii) radial flow v_r,

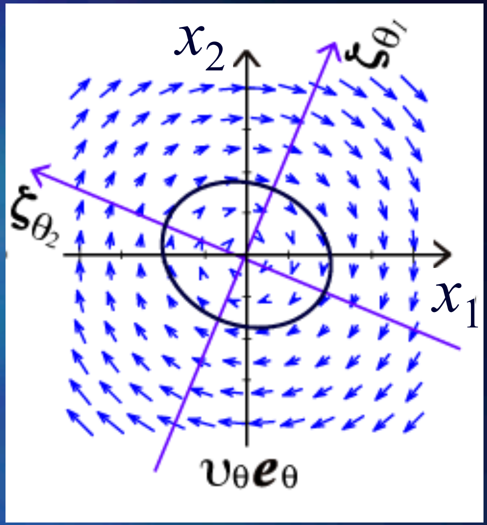
such as:

$\boldsymbol{v}' = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta$

$\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$: unit vectors of radial and azimuthal directions



3.1 Azimuthal flow



- v_θ : expressed as a specific quadratic form

$$v_\theta = \frac{1}{|\boldsymbol{x}'|}\, {}^t\boldsymbol{x}'\, \boldsymbol{Q}_\theta\, \boldsymbol{x}' \qquad \boldsymbol{x}' = (\, x_1, x_2 \,)$$

$$\nabla \boldsymbol{v} = \begin{pmatrix} a_{21} & -(a_{11}-a_{22})/2 \\ -(a_{11}-a_{22})/2 & -a_{12} \end{pmatrix}$$

$\boldsymbol{Q}_\theta$: unitary matrix having two real eigenvalues $\lambda\theta_i$ and their orthogonal eigenvectors $\zeta\theta_i$ $(i=1, 2)$.

- the feature of v_θ : $\lambda\theta_i$ specify

3.2 swirlity

- if $\lambda\theta_1$ and $\lambda\theta_2$ have the same sign
  - ➢ v_θ has the same direction around a point
  - ➢ swirling flow

Define swirlity ϕ *1 that represents the unidirectionality and intensity of v_θ ($\lambda\theta_i$) in terms of the geometrical average.

$$\phi := \mathrm{sgn}(\lambda\theta_1\ \lambda\theta_2)\ |\lambda\theta_1\ \lambda\theta_2|^{1/2}$$

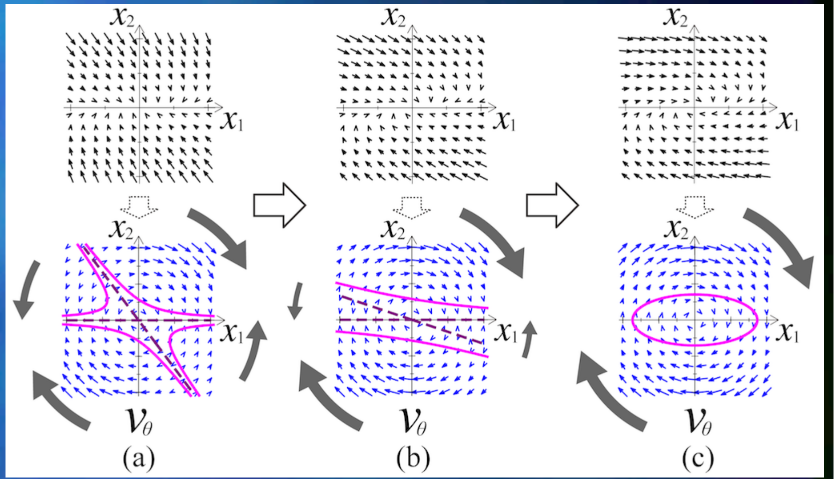


Fig. 3.1: local flow and decomposed v_θ in flow transition into a vortex. $0<\phi$ in (a) and (b), and $0<\phi$ in (c), i.e., vortical flow.

ϕ is defined in vortical/non-vortical flow. ⇨ applicable to prediction of a vortex *2

(*1: Nakayama, *Fluid Dyn. Res.*, 2014,
*2: Nakayama, *ICTAM2016*)

<Question>

- What is the physical interpretation of complex eigenvalues (imaginary part ψ) of $\nabla \boldsymbol{v}$?
- how to relate ψ (eigenvalues of 3-dim $\nabla \boldsymbol{v}$) to flow topology in a plane...

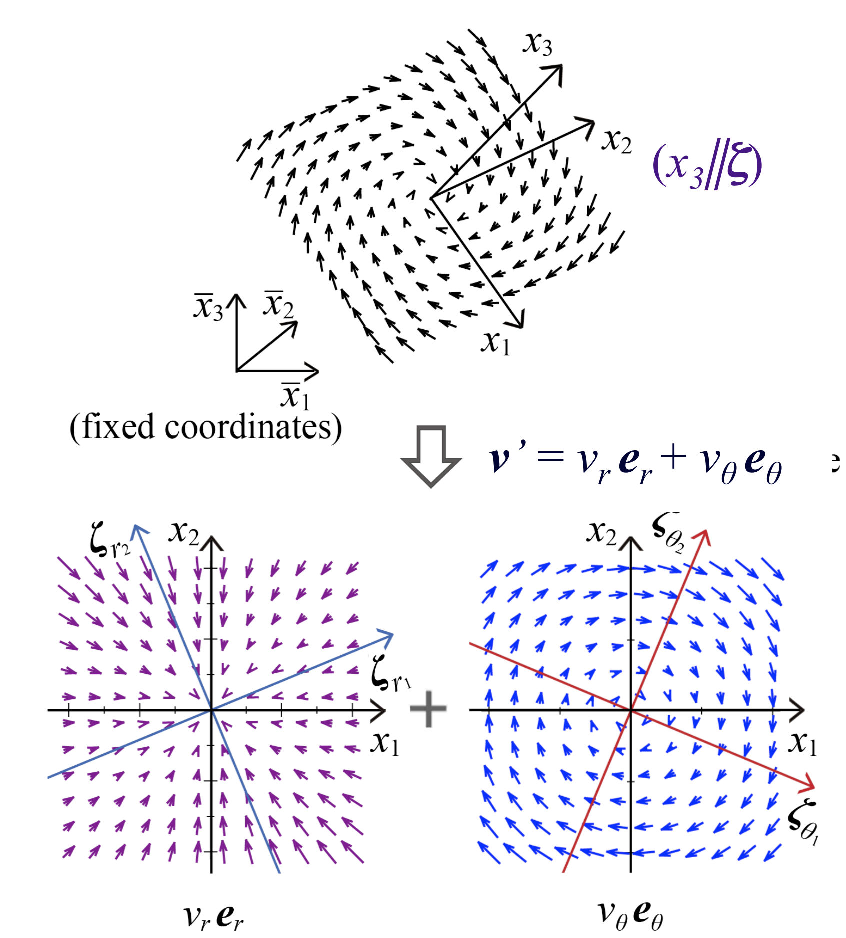
$$\boldsymbol{x} = 2\exp(\varepsilon_R t)\{\cos(\psi t)\boldsymbol{\xi}_{pl} - \sin(\psi t)\boldsymbol{\eta}_{pl}\} + \exp(\varepsilon_a t)\boldsymbol{\zeta}$$

$$\nabla \boldsymbol{v} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

3.3 Relationships between three dimensional eigenvalues and swirlity

- 3-dim $\nabla \boldsymbol{v}$ has at least one real eigenvalue ε_a and real eigenvector ζ.
- We define a coordinate system (x_i) where the x_1-x_2 plane with two orthonormal bases is an arbitrary plane linear independent of (non-parallel to) ζ (set as x_3 axis):
- $\nabla \boldsymbol{v}$ in this coordinate system:

$$\nabla \boldsymbol{v} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{11} & a_{12} & 0 \\ a_{31} & a_{32} & \varepsilon_a \end{pmatrix}$$



3.4 Invariance of swirlity

In this coordinate system, it is mathematically proven that

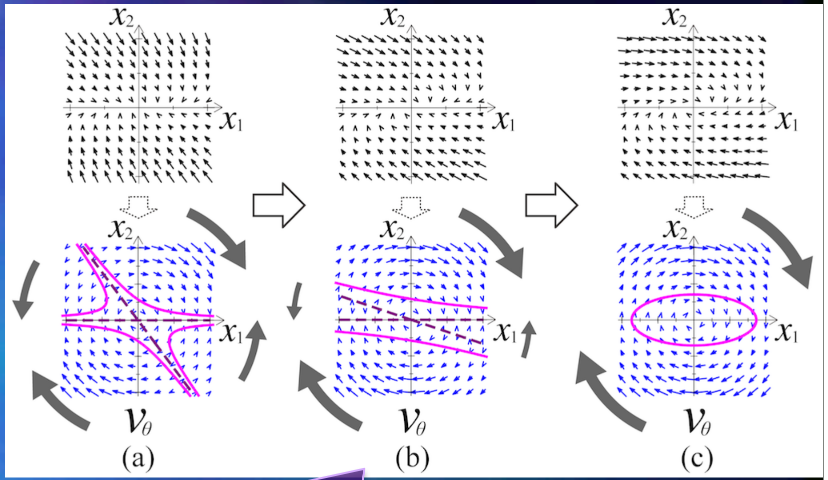
- $0 < |\lambda_{\theta_1} \lambda_{\theta_2}|$ i.e., $0 < \phi$ (topological condition)
- $\nabla \boldsymbol{v}$ has complex eigenvalues* (algebraic condition)

are equivalent. ϕ is expressed as;

- $\psi = \phi$ and $\phi = Q - 3\varepsilon_a{}^2$

➢ ψ in complex eigenvalues of $\nabla \boldsymbol{v}$: geometrical average of v_θ.

➢ swirlity is invariant independent of the arbitrary plane (non parallel to ζ axis).



Note: ϕ is defined in vortical/non-vortical flow.

* $\Delta = (Q/3)^3 + (R/2)^2 > 0$ (Q, R: the 2nd and 3rd invariants of $\nabla \boldsymbol{v}$)
Δ definition(Chong et al., *Phy. Fluids*, 1990)

(Nakayama, *Fluid Dyn. Res.*, 2014)

3.5 radial flow v_r

- similar to v_θ, v_r is expressed as a specific quadratic form:

$$v_r = \frac{1}{|\boldsymbol{x}'|}\,{}^t\boldsymbol{x}'\,\boldsymbol{Q}_r\,\boldsymbol{x}'$$

$$\nabla \boldsymbol{v} = \begin{pmatrix} a_{11} & (a_{21}+a_{12})/2 \\ (a_{21}+a_{12})/2 & a_{22} \end{pmatrix}$$

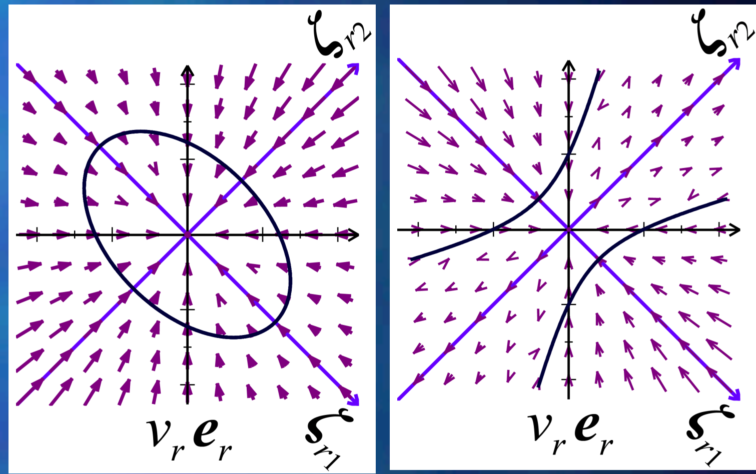


Fig. 3.2: decomposed v_r where (left) $0<\sigma$ ($\lambda r_i <0$) and (right) $\sigma<0$ (with same complex eigenvalues of $\nabla \boldsymbol{v}$).

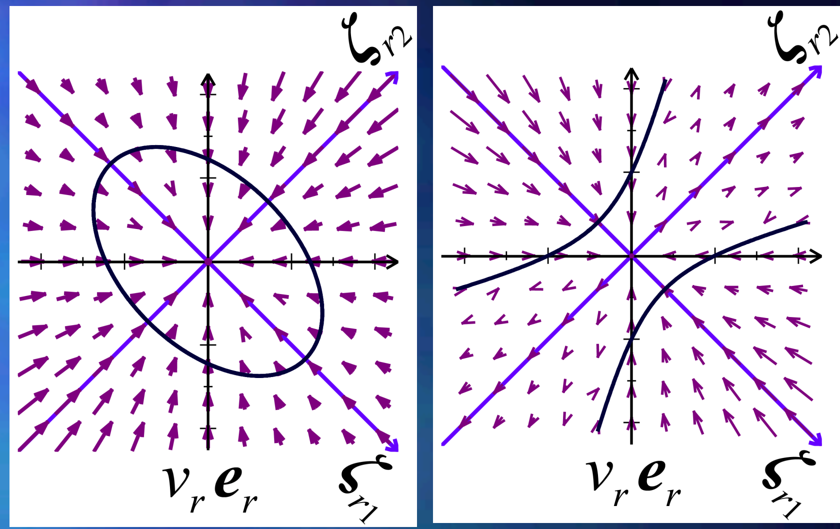
- $\boldsymbol{Q}_r$: unitary matrix having two real eigenvalues λr_i and their orthogonal eigenvectors ζr_i (i=1, 2).

3.6 sourcity

- if λ_{r_1} and λ_{r_2} have the same sign
  ($0 < |Q_r|$, Q_r: unitary)
  - v_r has the same direction around a point
- complete inflow from all directions
  - $\lambda_{r1}, \lambda_{r_2} < 0$

Define sourcity σ *1 that represents the unidirectionality and intensity of v_r (λ_{r_i}) in terms of the geometrical average

$$\sigma := \mathrm{sgn}(\lambda_{r_1} \lambda_{r_2}) \, |\lambda_{r_1} \lambda_{r_2}|^{1/2}$$



(Fig. 3.2: decomposed v_r where (left) $0 < \sigma$ ($\lambda_{r_i} < 0$) and (right) $\sigma < 0$ (with the same complex eigenvalues of $\nabla \boldsymbol{v}$).)

(Nakayama, *Fluid Dyn. Res.*, 2014)

3.7 another invariant of $\nabla \boldsymbol{v}$

If $\nabla \boldsymbol{v}$ (=A) has conjugate complex eigenvalues ($\varepsilon_R \pm i\psi$),

- Eigenequation for the complex eigenvalues:

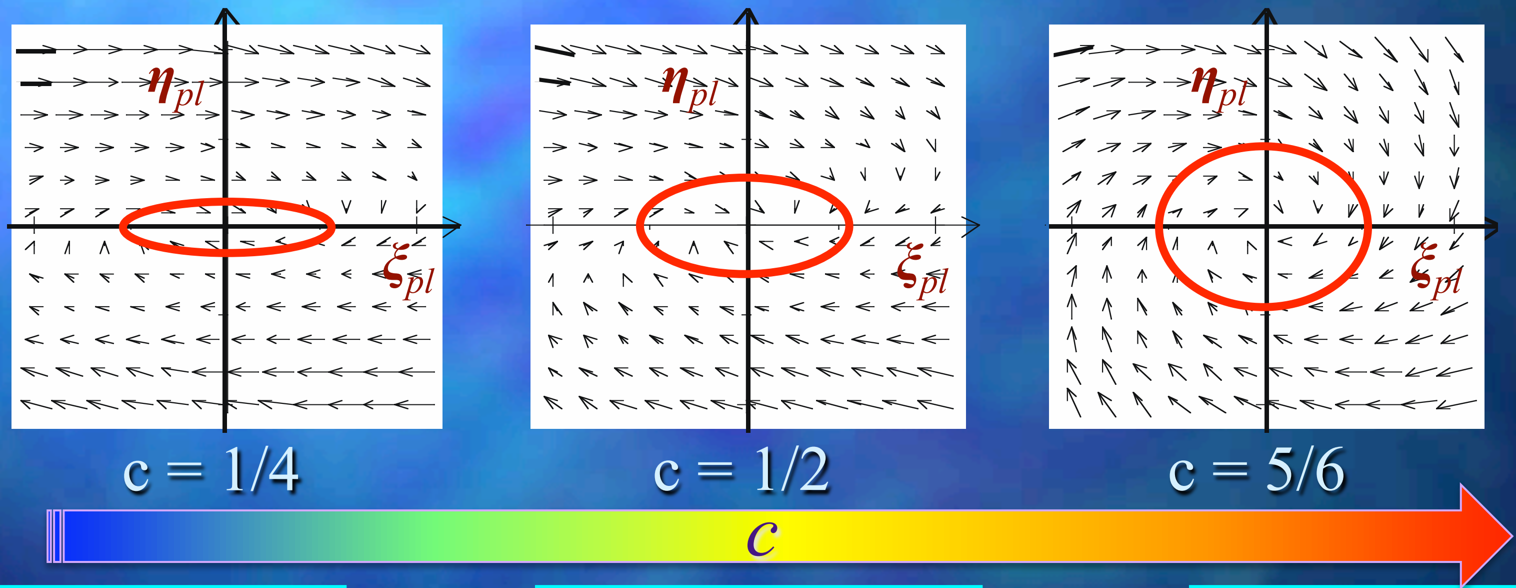
$$A(\boldsymbol{\xi}_{pl} \pm i\boldsymbol{\eta}_{pl}) = (\varepsilon_R \pm i\psi)\,(\boldsymbol{\xi}_{pl} \pm i\boldsymbol{\eta}_{pl})$$

$$(A = [a_{ij}] = [\,\partial v_i / \partial x_j\,])$$

- differently from the real eigenvector, $\boldsymbol{\xi}_{pl}$ and $\boldsymbol{\eta}_{pl}$ are restricted in terms of the ratio of their norms (lengths).
- $c = |\boldsymbol{\xi}_{pl}| \,/\, |\boldsymbol{\eta}_{pl}|$
- c is an invariant quantity.
- note that $\boldsymbol{\xi}_{pl}$ and $\boldsymbol{\eta}_{pl}$ can be set as $\boldsymbol{\xi}_{pl} \perp \boldsymbol{\eta}_{pl}$.

3.8 symmetry quantity of vortical flow

- c has not been considered in the topological analysis.
- the topology depends on c, and c represents the flow symmetry.



$c \approx 0$ much skewed	$c = 0.5$ elliptic	$c = 1$ axisymmetric

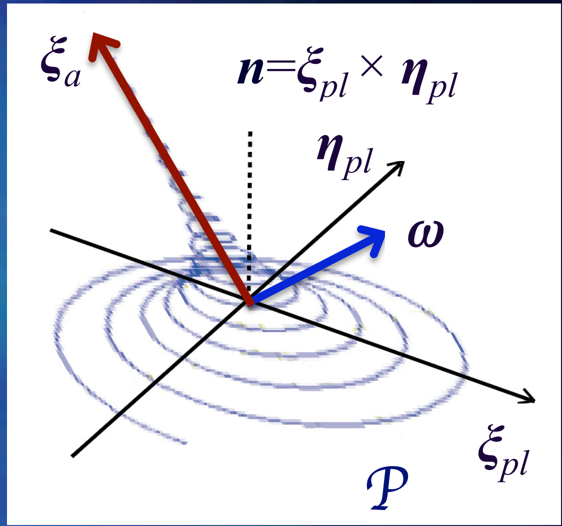
Fig. 3.3: flow geometry in ξ_{pl}-η_{pl} plane with **same complex eigenvalues** $(\varepsilon_R, \psi) = (-1, 2)$

3.9 vortex space

- We define "vortex space" V where the orthonormal bases are parallel to:

$$\boldsymbol{\xi}_{pl}\,,\ \boldsymbol{\eta}_{pl}\ \ (\boldsymbol{\xi}_{pl} \perp \boldsymbol{\eta}_{pl}),\ \boldsymbol{\xi}_{pl}\times\boldsymbol{\eta}_{pl}$$

- $\nabla \boldsymbol{v}$ in the coordinate system of V:



(Fig. 2.1: vortical flow topology ($\varepsilon_R < 0$))

$$\nabla \boldsymbol{v} = \begin{pmatrix} \varepsilon_R & c\psi & \omega_2 \\ -\psi/c & \varepsilon_R & -\omega_1 \\ 0 & 0 & \varepsilon_a \end{pmatrix}$$

$\omega_1 = (\boldsymbol{\omega},\ \boldsymbol{\xi}_{pl}/|\boldsymbol{\xi}_{pl}|)$

$\omega_2 = (\boldsymbol{\omega},\ \boldsymbol{\eta}_{pl}/|\boldsymbol{\eta}_{pl}|)$

$\omega_3 = -(c+1/c)\psi$ (ω_3 : eigen-helicity−density (Zhang et al., *Phy. Fluids*, 2006)

The vortex space facilitates to investigate :

- detail topology
- physical features such as pressure minimum or vortex stretching

3.10 Flow feature in swirl plane and physical quantities

characteristics of v_r and v_θ in $\mathcal{P}$:

$$\lambda r_1, \lambda r_2 = \varepsilon_R \pm |c - 1/c|\phi/2$$

$$\lambda\theta_1, \lambda\theta_2 = -c\phi, -\phi/c$$

- swirlity and sourcity

$$\phi = \psi = |\lambda\theta_1 \lambda\theta_2|^{1/2}$$

$$\sigma = \mathrm{sgn}(\alpha)\,|\alpha|^{1/2}$$

$$\alpha = \varepsilon_R^2 - (c - 1/c)^2 \phi^2/4$$

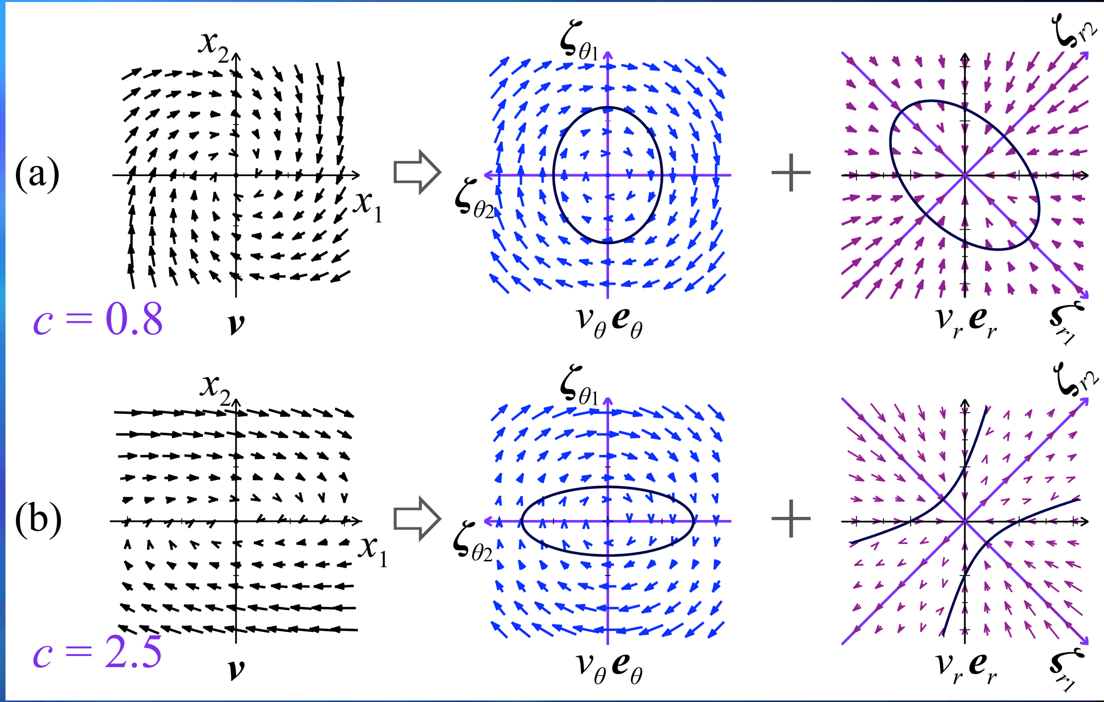


Fig. 3.4: flow geometry in $\mathcal{P}$ with **same complex eigenvalues** $(\varepsilon_R, \psi) = (-1, 2)$ but different c where (a) c=0.8 (σ = 0.9), and (b) c=2.5 (σ = −1.9).

(Nakayama, *Fluid Dyn. Res.*, 2014)

<Question>

- What is the physical interpretation of complex eigenvalues (real part ε_R) of $\nabla \boldsymbol{v}$?

$$\boldsymbol{x} = 2\exp(\varepsilon_R t)\{\cos(\psi t)\boldsymbol{\xi}_{pl} - \sin(\psi t)\boldsymbol{\eta}_{pl}\} + \exp(\varepsilon_a t)\boldsymbol{\zeta}$$

$$\nabla \boldsymbol{v} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

3.11 Insufficiency of ε_R as classifying inflow/outflow vortices

$$\nabla v = \begin{pmatrix} \varepsilon_R & c\psi & \omega_2 \\ -\psi/c & \varepsilon_R & -\omega_1 \\ 0 & 0 & \varepsilon_a \end{pmatrix}$$

$$\lambda r_1, \lambda r_2 = \varepsilon_R \pm |c - 1/c|\psi/2$$

- Radial flow in $\mathcal{P}$ (in vortical flow):

$$\frac{\lambda r_1 + \lambda r_2}{2} = \varepsilon_R \quad \text{(invariant)}$$

- ε_R is <u>difficult to distinguish complete inflow(outflow) or mixed flow with both inflow and outflow</u>.

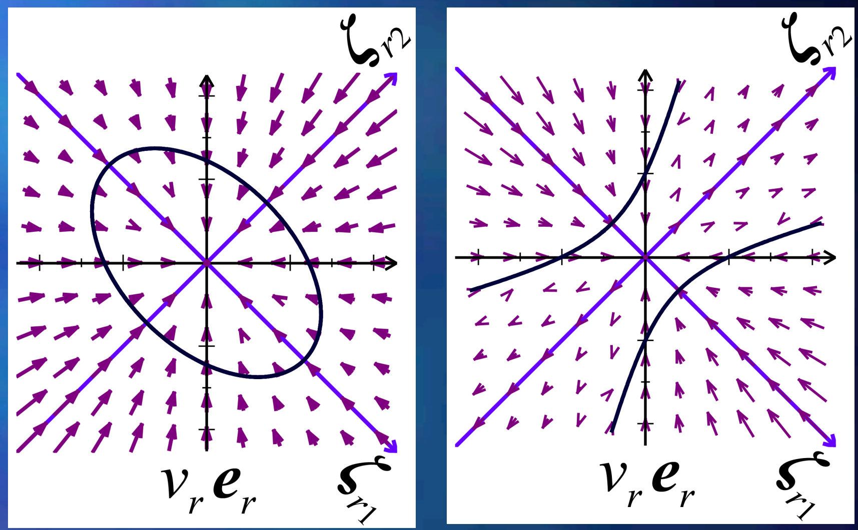


Fig. 3.5: radial flow with **same complex eigenvalues** $(\varepsilon_R, \psi) = (-1, 2)$ but different c where (a) c=0.8 ($\sigma = 0.9$), and (b) c=2.5 ($\sigma = -1.9$).
(same as Fig. 3.2)
(Nakayama, *Fluid Dyn. Res.*, 2014)

3.12 example of inflow vortices classified by ε_R (<0)

- An example in an isotropic homogeneous turbulence:

- about 85% of inflow vortices classified by $\varepsilon_R < 0$ are <u>vortices with mixed inflow and outflow</u>.
- This fact gives an important feature in the effect of vortex stretching.

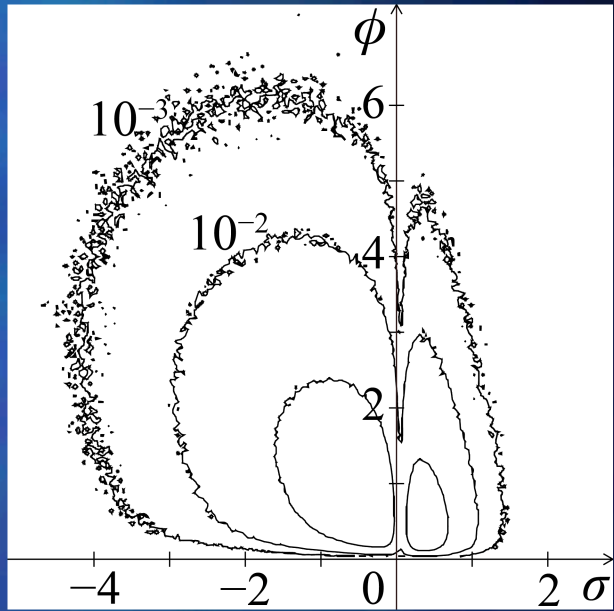


Fig.3.6 JPDF (Joint Probability Density Function) of σ and ϕ in terms of $\varepsilon_R < 0$ (average-inflow vortices).

(Nakayama, *Phy. Rev. Fluid*, 2017)

3.13 Relationships between swirlity and vortical flow symmetry

(vortices in an isotropic homogeneous turbulence)

- ϕ and c have high correlation.
- flow symmetry is important for development of a vortex.

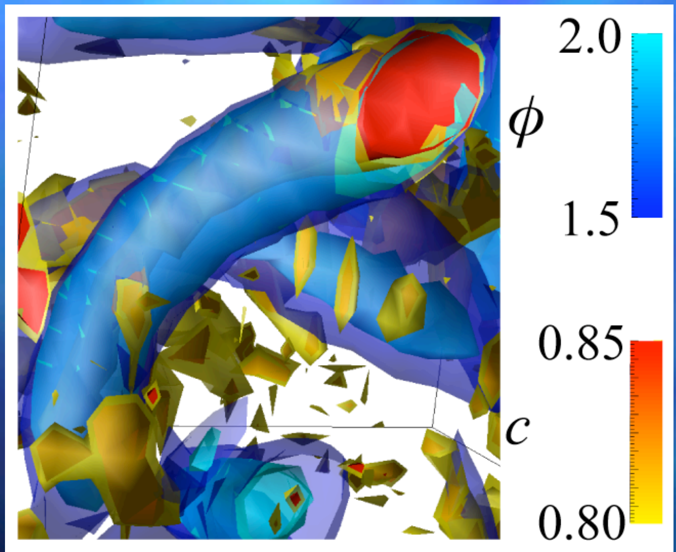


Fig. 3.7 : contours of ϕ and c.

ϕ and c codevelop or codecay.

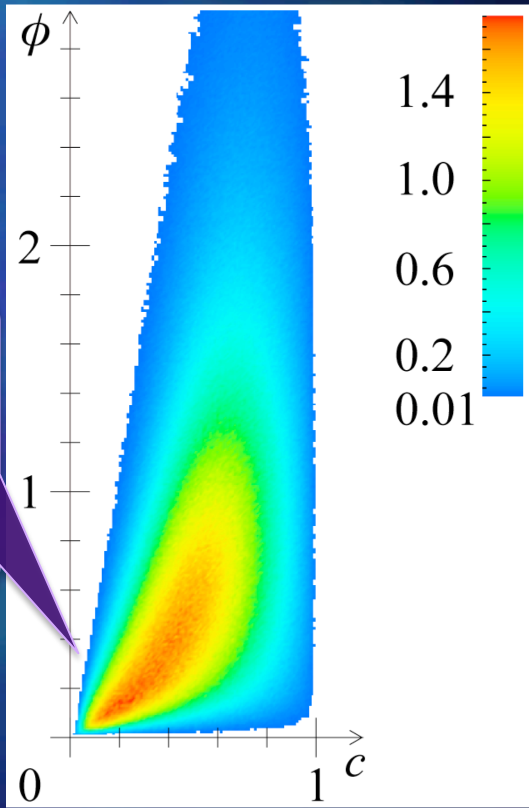
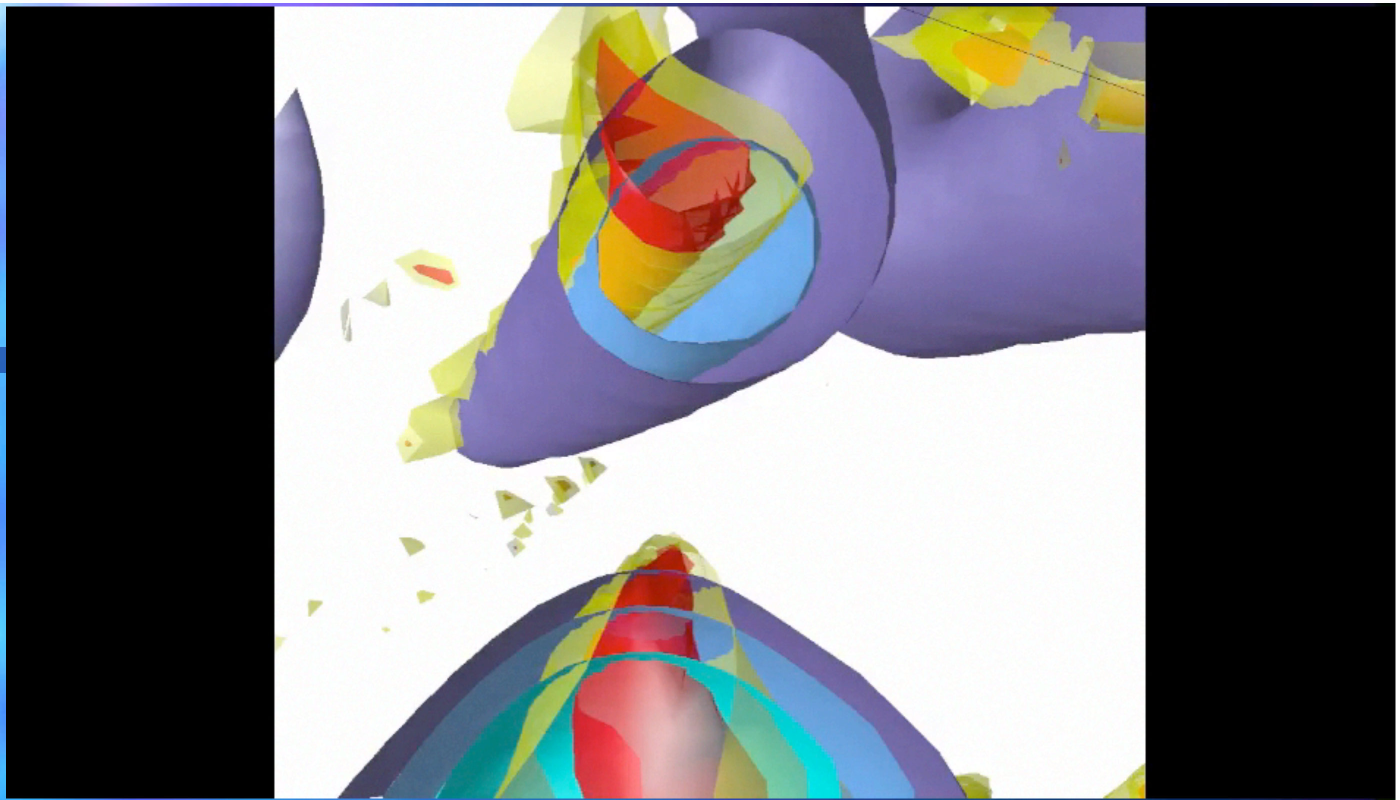


Fig.3.8 JPDF of c and ϕ.

(ref. e.g., Nakayama, *Phy. Rev. Fluids*, 2017)

3.14 animation of a vortex

(a vortex in an isotropic homogeneous turbulence)



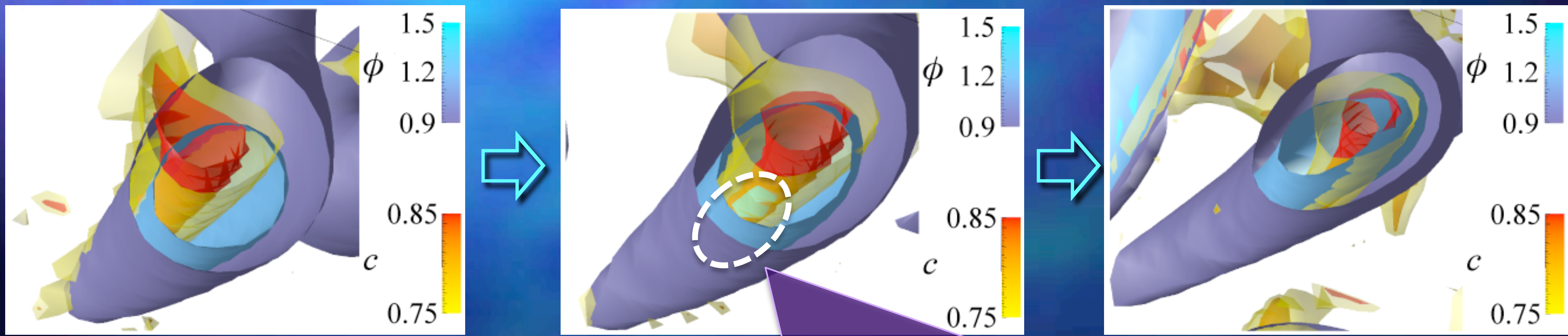


Fig. 3.9 : animation of a vortex with contours of ϕ and c.

When c increases, a new contour of ϕ (intense ϕ region) appears.

4. Physical feature of a vortex

Important physical features of vortex associated with the topology:

- Pressure minimum
- vortex stretching

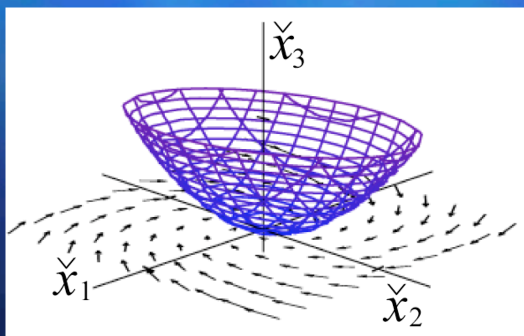
Vortex space facilitates the analysis of these features.

4.1 Pressure minimum by topology

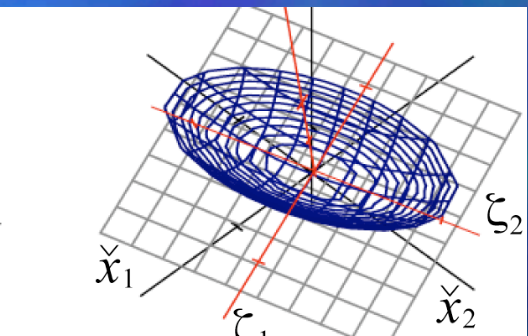
- We focus on the Hesse matrix $\boldsymbol{H} = [h_{ij}] = [-p,_{ij}/\rho]$ of the pressure by differentiating the Navier-Stokes equation,
- discarding unsteady strain and viscous terms, to estimate the pressure minimum derived from the vortical motion[*1].

➢ $\boldsymbol{H} = (AA + {}^tA{}^tA)/2 \quad (A = \nabla \boldsymbol{v})$ [*2]

☐ the pressure min. by vortical flow in $\mathcal{P}$ should be estimated.



pressure feature in $\mathcal{P}$



pressure minimum plane

(*1: Jeong et al., *J. Fluid Mech.*, 1995, *2: Nakayama et al., *Fluid Dyn. Res.*, 2014)

4.2 Pressure minimum in swirl plane

- In vortex space $\boldsymbol{H}$ is expressed as:

$$\boldsymbol{H} = \left(\begin{array}{cc|c} \varepsilon_R{}^2\varphi\psi^2 & (c-1/c)\varepsilon_P\psi & \iota_1 \\ (c-1/c)\varepsilon_R\psi & \varepsilon_R{}^2-\psi^2 & \iota_2 \\ \hline \iota_1 & \iota_2 & \varepsilon_a{}^2 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} \lambda_{p_1} & 0 & \iota_1' \\ 0 & \lambda_{p_2} & \iota_2' \\ \iota_1' & \iota_2' & \varepsilon_a{}^2 \end{pmatrix}$$

By rotating the bases in $\mathcal{P}$ in accordance with the orthonormal eigenvectors of the Hessians λ_{p_i} in $\mathcal{P}$.

$$\lambda_{p_i} = (\varepsilon_R{}^2-\psi^2) \pm |(c-1/c)\varepsilon_R|\psi$$

Pressure minimum condition: $\lambda_{p_1}, \lambda_{p_2} < 0$ $\Rightarrow c|\varepsilon_R| < \psi \ \ (\psi = \phi)$

c relates to the pressure minimum in $\mathcal{P}$.

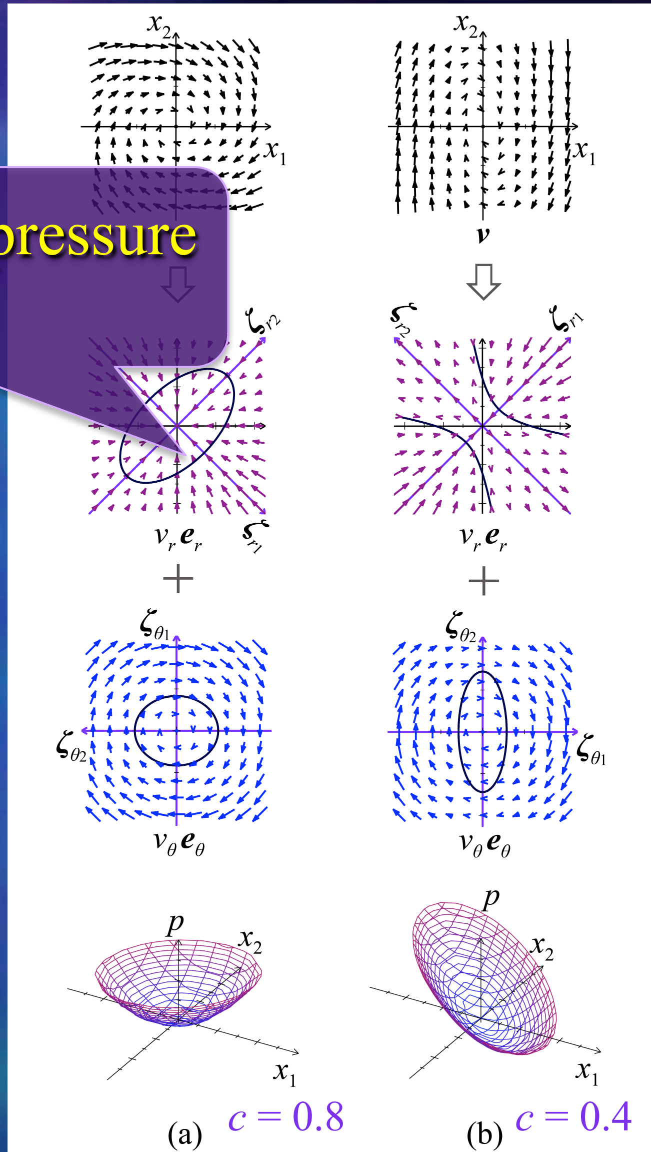
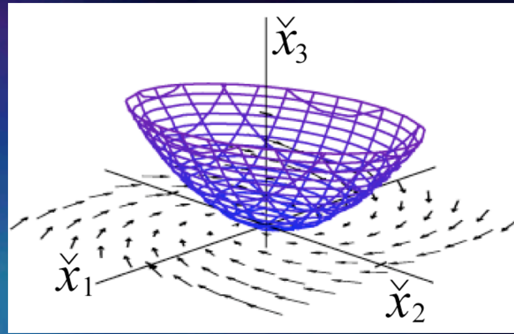


Fig. 4.1: flow geometry in $\mathcal{P}$ and pressure minimum feature where $(\varepsilon_R, \psi) = (-1, 2)$, and (a) c=1.2, and (b) c=0.4.

4.3 on universal definition



- The topology in $\mathcal{P}$ and resulting pressure minimum is related.
- The vortex definition with this criteria unifies (satisfies) the major vortex definitions (Nakayama et al., *Fluid Dyn. Res.*, 2014) :

1. Δ definition (Chong et al., *Phy. Fluids*, 1990) :
   - $\nabla \boldsymbol{v}$ has complex eigenvalues
2. Q definition (Hunt et al., CTR-S88 1988) :
   - vorticity exceeds rate of strain
   - 3-dim pressure Laplacian
3. λ_2 definition (Jeong et al., *J. Fluid. Mech.*, 1995) :
   - existence of pressure min. plane by vortical flow

approaching the universal definition of a vortex...

4.4 vortex stretching

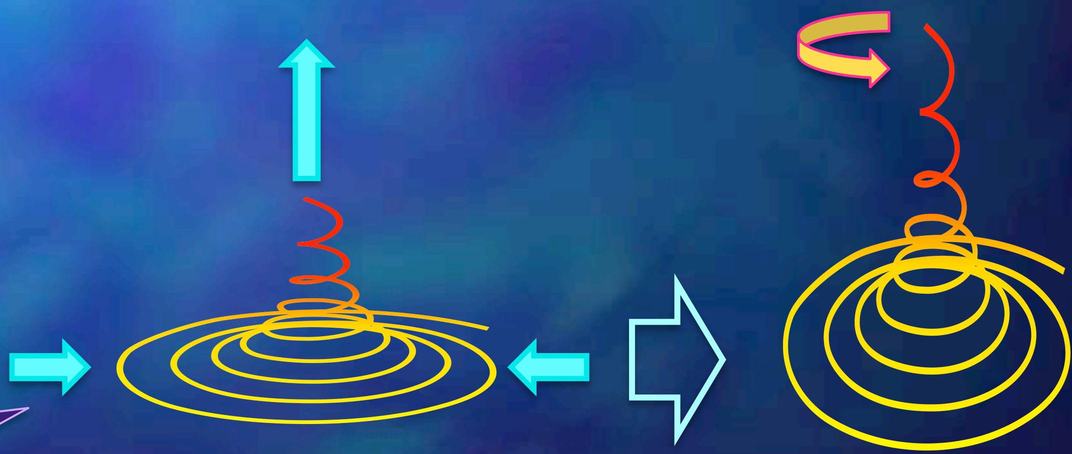
- strengthening the vorticity with strain (compression/tension)
- strain: the rate of strain tensor (its eigenvalues (eigenvectors))

  $s_{ij} = (\partial v_i / \partial x_j + \partial v_j / \partial x_i) / 2$

- vortex stretching rate δ, i.e., the rate of generation of enstrophy $|\boldsymbol{\omega}|^2$ is expressed as (Jimenez, *J. Fluid. Mech.*, 1993):

$$\delta = \omega_i s_{ij} \omega_j \ / \ |\boldsymbol{\omega}|^2$$
$$= \omega_i s_{ij} \omega_j \ / \ (\omega_i \omega_i)$$

However, eigenvectors of s_{ij} are not identical to the swirl plane



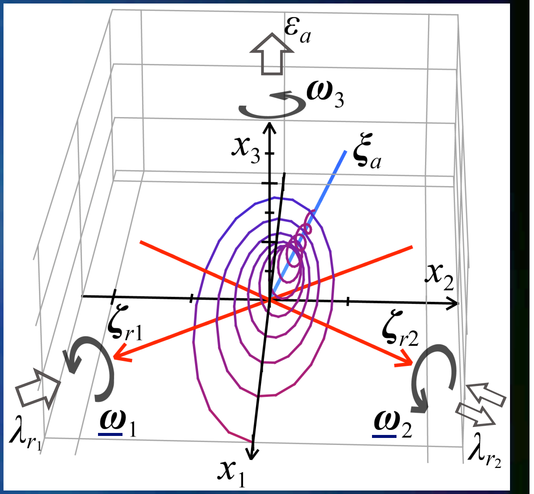
4.5 formulation of vortex stretching

- In the vortex space, rotating the bases in $\mathcal{P}$ in accordance with ξr_i (eigenvector of λr_i) (rotating $\pi/4$) gives :

$$\delta = \{ \lambda r_1 \underline{\omega}_1^2 + \lambda r_2 \underline{\omega}_2^2 + \lambda r_3 \omega_3^2 \} / (\omega_i \omega_i)$$

$$\underline{\omega}_1 = (\omega_1 - \omega_2)/\sqrt{2},\ \underline{\omega}_2 = (\omega_1 + \omega_2)/\sqrt{2}$$

$$(\lambda r_3 = \varepsilon_a,\ \omega_3 = -(c+1/c)\phi)$$



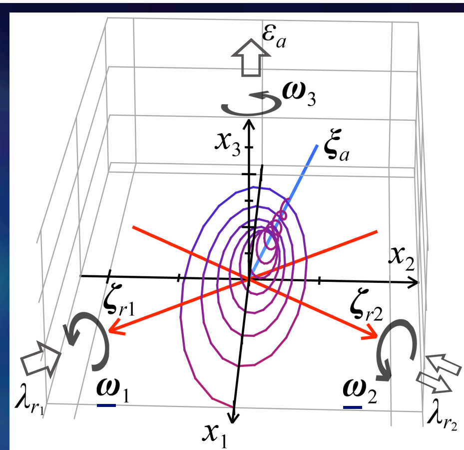
analyse vortex stretching by decomposition of vorticity components parallel and normal to P .

- $0 < \sigma$: increases effectively both swirl and axis orthogonality
- $\sigma < 0$: increase swirl but decreases the orthogonality

➢ λr_i and σ specify the characteristic of the stretching

4.6 effect of vortex stretching

- inflow in all directions **effectively strengthens swirl** (ω_3), and **increases orthogonality of a vortical axis**.
- This characteristic is specified by sourcity.
- Vorticity in the vortex stretching can be characterized by <u>decomposition of vorticity in terms of components parallel and normal to $\mathcal{P}$</u>.



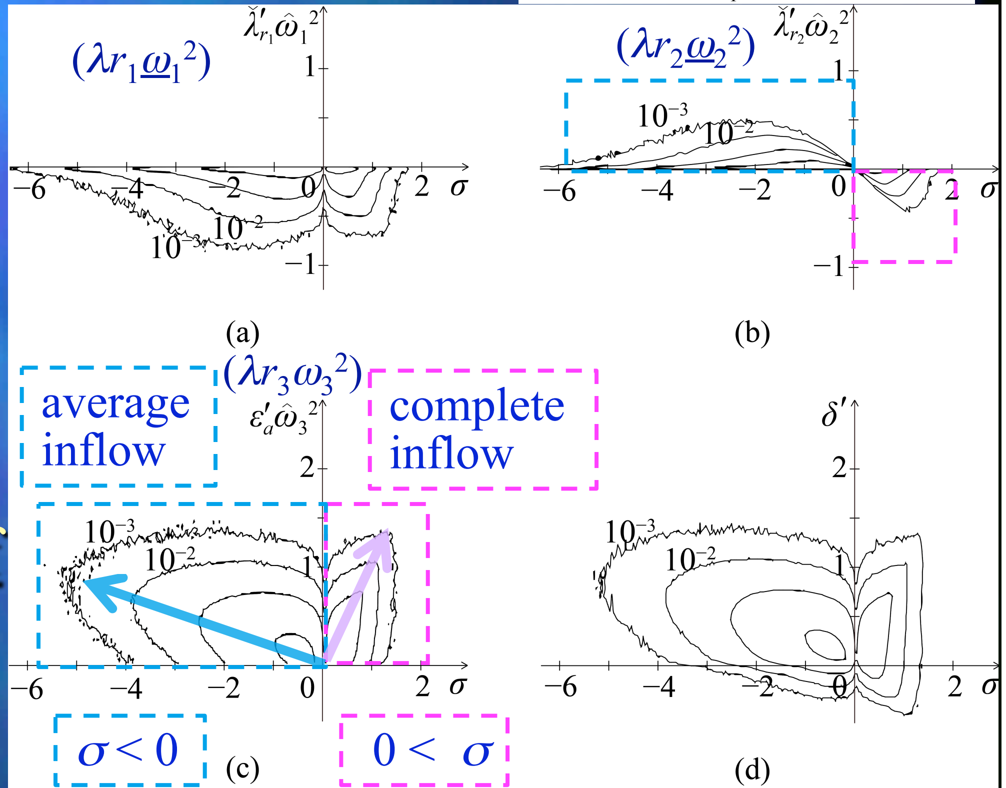


Fig. 4.2: each term of the vortex stretching in average-inflow vortices (ε_R<0), non-dimensionalized by the root mean square value of the vorticity in an isotropic homogeneous turbulence.

(ref.: Nakayama, *Phy. Rev. Fluids*, 2017)

4.7 characteristic of vortex stretching

$$\delta = \lambda_{r1}\underline{\omega}_1^2 + \lambda_{r2}\underline{\omega}_2^2 + \lambda_{r3}\omega_3^2$$

A life of a vortex in an isotropic homogeneous turbulence.

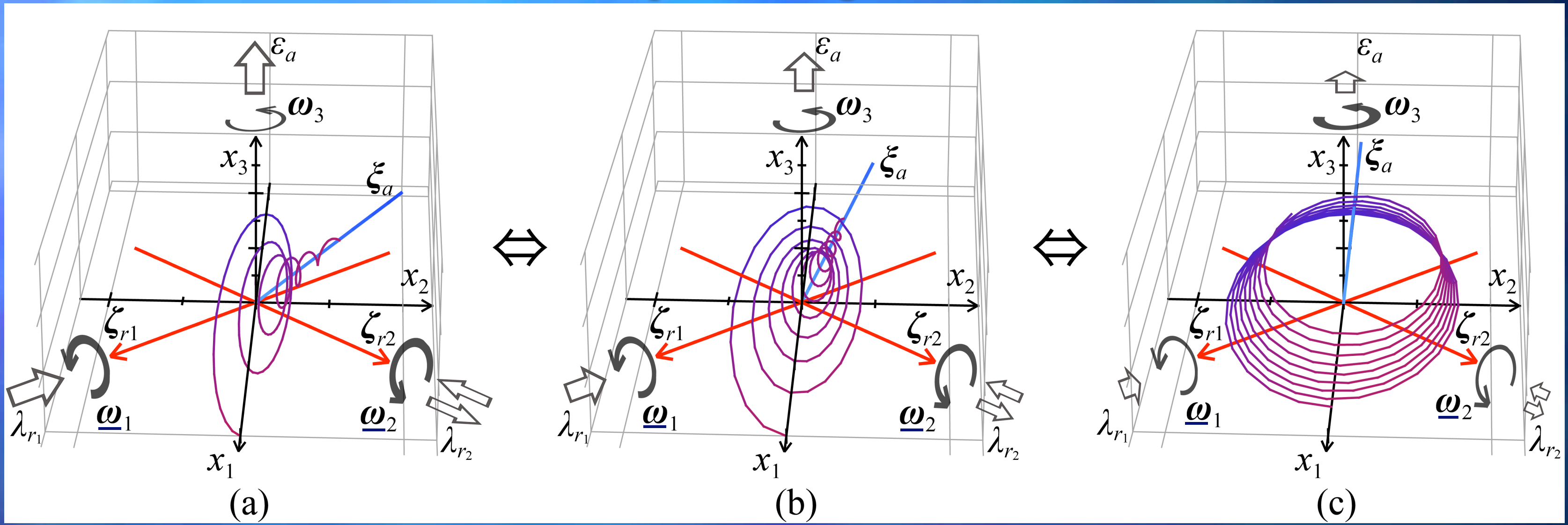


Fig. 4.3 : A life of an average inflow vortex (ε_R<0) in an isotropic homogeneous turbulence (generation ➡development ➡decay).

- $0 < \sigma$: increases effectively both swirl and axis orthogonality
- $\sigma < 0$: increase swirl but decreases the orthogonality

➢ λr_i and σ specify the characteristic of the stretching

5. A definition of a vortical axis

5.1 Focusing on the last eigenvector of $\nabla \boldsymbol{v}$

We go back to the local flow topology where $\nabla \boldsymbol{v}$ has complex conjugate and real eigenvalues, $\varepsilon_R \pm i\psi$ and ε_a, and their respective eigenvectors $\boldsymbol{\xi}_{pl} \pm i\, \boldsymbol{\eta}_{pl}$ and $\boldsymbol{\zeta}$,

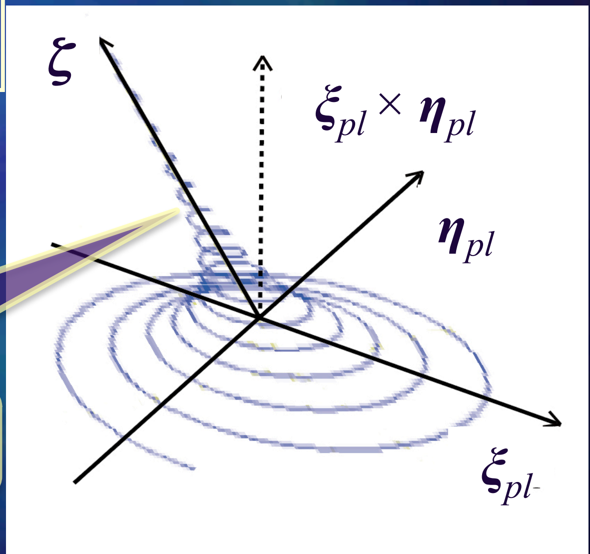
➔ Flow trajectory:

$$\boldsymbol{x} = 2\exp(\varepsilon_R t)\{\cos(\psi t)\boldsymbol{\xi}_{pl} - \sin(\psi t)\boldsymbol{\eta}_{pl}\} + \exp(\varepsilon_a t)\boldsymbol{\zeta}$$

Flow in $\mathcal{P}$ has been examined.

➔ We focus on a real eigenvector.

Flow proceeds (converges) along $\boldsymbol{\zeta}$.

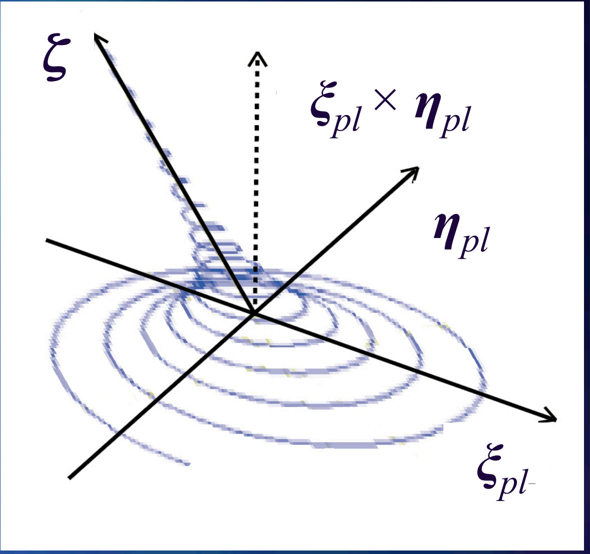


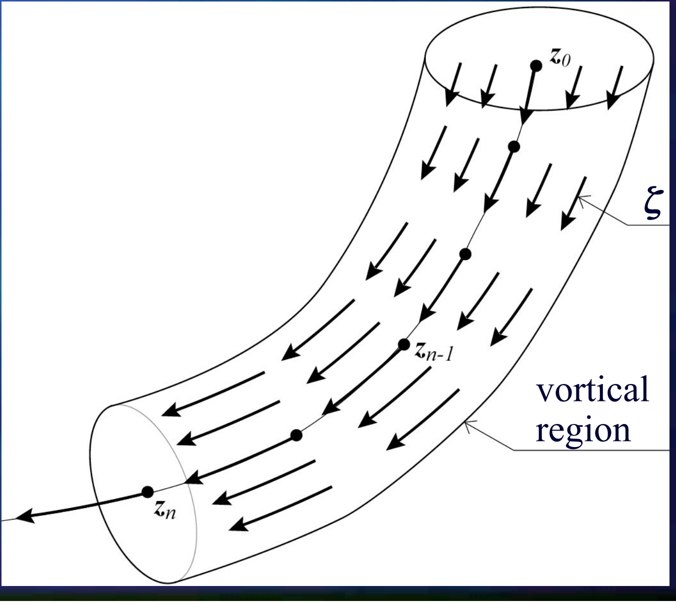
(Fig. 2.1: vortical flow topology ($\varepsilon_R < 0$))

5.2 Eigen-vortical-axis Line

- ζ indicates an axis direction in terms of the invariant local topology, and can be defined in vortical region V where $0 < \phi$. (∇v has complex eigenvalues)
- Define a vortical axis along ζ
- Eigen-vortical-axis Line $\alpha(x)$ ($\alpha = [\alpha_i]$ (i=1,2,3)) in V such that

$$\frac{d\alpha_1}{\zeta_1} = \frac{d\alpha_2}{\zeta_2} = \frac{d\alpha_3}{\zeta_3}$$





(5.3 Relationship between eigen-vortical-axis line and vorticity line)

<Question> (maybe later)

- What is the relationships between vorticity vector and eigen-vortical-axis?

The relationships between ζ and ω can be formulated, relating to the topology, but they are not simple.

5.4 Analysis of eigen-vortical-axis/vorticitiy lines in isotropic homogeneous decaying turbulence

- Pseudo Spectral Method with phase-shift method

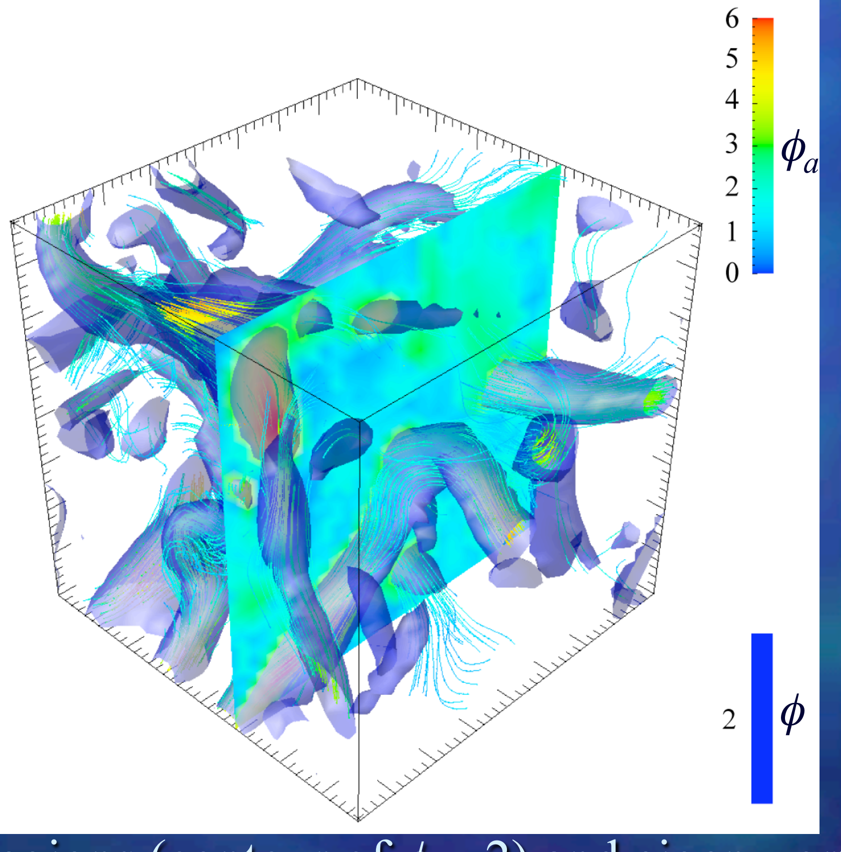


Fig.5.1: Vortical regions (contour of $\phi = 2$) and eigen-vortical-axis line (bold line)/ vorticity line (narrow line) in a sub-domain ($133\eta \times 133\eta \times 43\eta$). ($\phi_a$: ϕ in axes)

5.5 feature of a traced axis –consistency with vortical core region

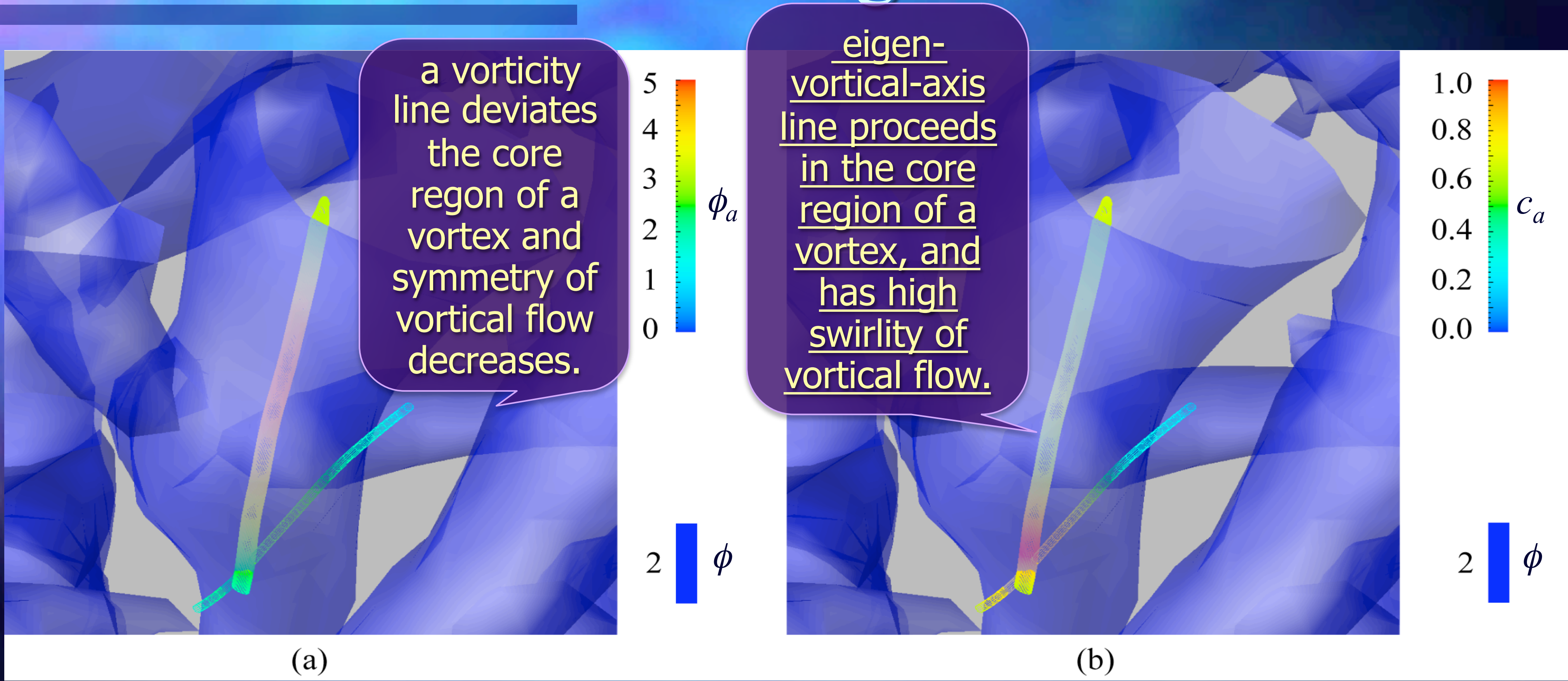
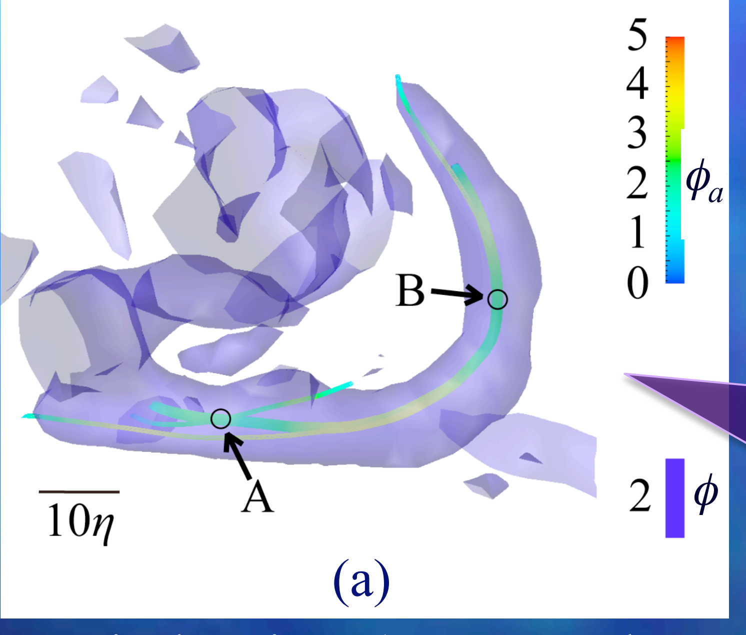


Fig. 5.2: Zoomed vortical regions (ϕ=2) and vortical axes traced by a eigen-vortical-axis line (bold line) and a vorticity line (narrow line), where the color in the axes shows (a) ϕ (ϕ_a) and (b) c (c_a) in the axes.

5.6 another example of traced axes

- eigen-vortical-axis line (EVAL) passes the core region.



EVAL follows the core region of a vortex.
A vorticity line that passes point A deviates the core region of a vortex with low swirlity.

Fig. 5.3: Vortical regions (contours where $\phi = 2$), and an eigen-vortical-axis line (bold line) and two vorticity lines (narrow lines) in a sub-domain (Kolmogorov length $\eta = 0.012$).

5.7 pressure minimum feature of a traced axis

- eigen-vortical-axis line has also pressute minimum, while a vorticity line does not have this feature.

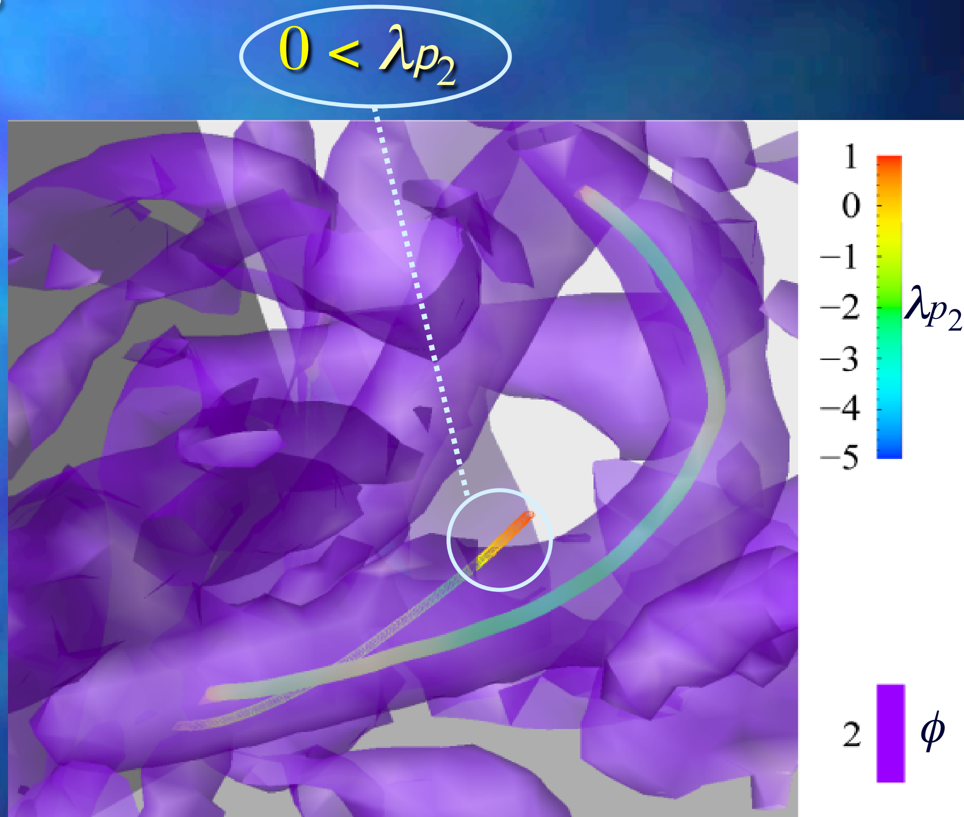


Fig. 5.4: Vortical axes traced by eigen-vortical-axis line (bold line) and vorticity line (narrow line).

5.8 Bundle features of eigen-vortical-axis/vorticitiy lines in isotropic homogeneous turbulence

- eigen-vortical-axis line tends to concentrate and have intense swirlity in the core region of vortices.

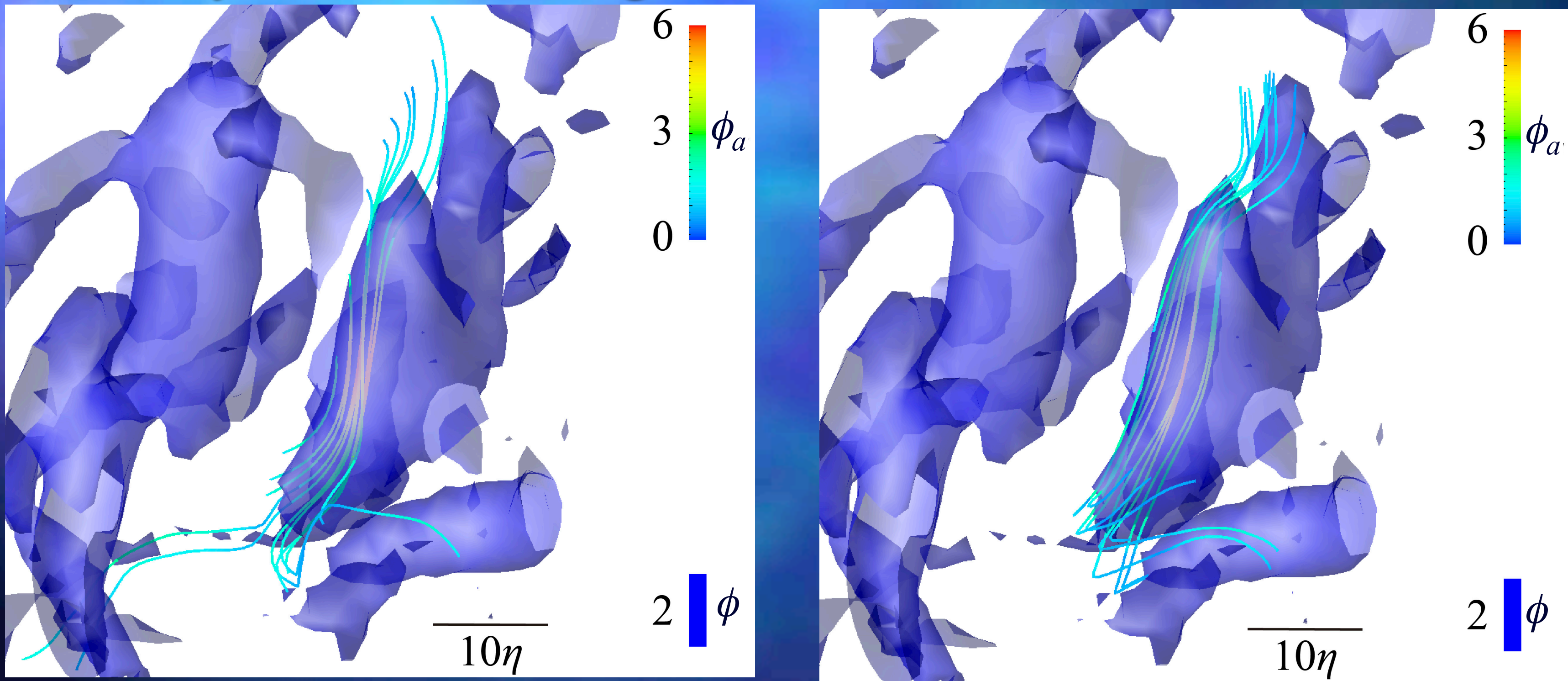
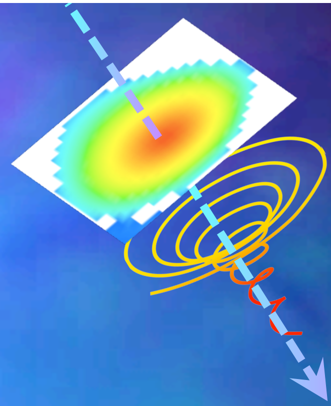


Fig. 5.5: bundle features of eigen-vortical-axis line (left)/vorticity line (right).
(ϕ_α, ω_a : ϕ and $|\omega|$ in axes)

6. Conclusion

1. Eigenvalues of the velocity gradient tensor are insufficient
   – to specify the detail flow topology
   – to relate the topology to physical characteristics of a vortex
2. Pressure minimum in the swirl plane and vortex stretching are specified by the detail topological quantities.
3. Vorticity should be decomposed into components parallel and normal to the swirl plane for specifying the vortex stretching.

(6. Conclusion)



4. A vortical axis with intense swirling might be along the local flow axis.
5. Vorticity and rate of strain tensor are important quantities, however, the behavior of present topological quantities contributes the vorticity.
6. The present topological approach enables us to watch flow with a new sight, i.e., detail flow characteristics.