# Hydrodynamic stability analysis in terms of action-angle variables

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# 1. Introduction

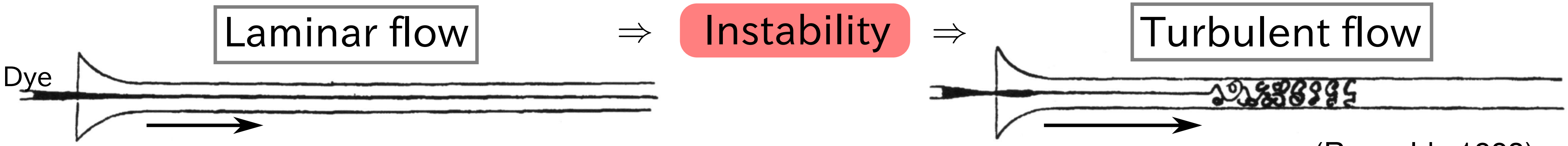
## How is Fluid mechanics different from “Classical” mechanics?

- Infinite degree of freedom
- Eulerian description for velocity field
  (vs. Lagrangian description for particle orbit)

⇒ The partial differential equations (PDEs) for vector and scalar fields

  - Mathematical methods used in classical mechanics are not directly applicable.
  - It is hard to solve the PDEs even by using computers.

⇒ Linear and weakly nonlinear stability analysis (perturbation analysis) is one of the feasible approaches.



Laminar flow ⇒ Instability ⇒ Turbulent flow

(Reynolds 1883)

# Outline

$\boxed{\text{Stability theory for Hamiltonian systems}} \xrightarrow{?} \boxed{\text{Hydrodynamic stability theory}}$

Based on the Hamiltonian viewpoint of fluid mechanics, the variational method is shown to be useful for predicting stability (both theoretically and numerically).

1. Introduction

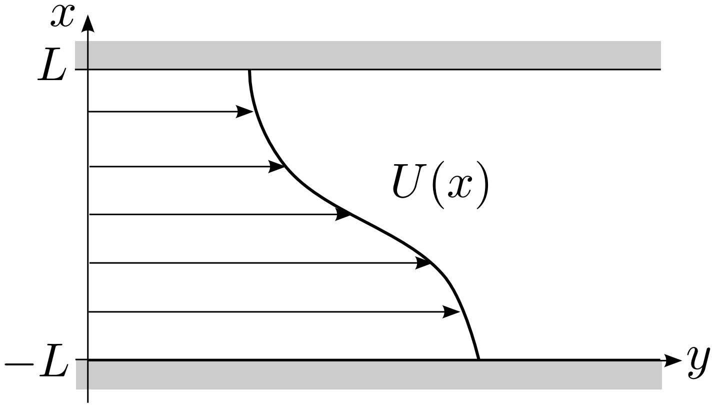
   (As a typical and simple hydrodynamic stability problem, we consider)
2. Rayleigh equation — stability of inviscid parallel shear flow
3. Action-angle variables (in classical mechanics)
4. Variational stability conditions (in classical mechanics)
5. Wave action (action variable in fluid mechanics)
6. Variational stability conditions for Rayleigh equation
7. Summary

# Rayleigh equation ~ Stability of inviscid parallel shear flow ~

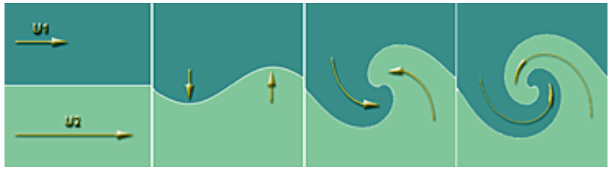
Shear flow $\boldsymbol{U} = U(x)\boldsymbol{e}_y$,
Disturbance $\tilde{\boldsymbol{u}} = \nabla[\phi(x)e^{ik(y-ct)} + \mathrm{c.c.}] \times \boldsymbol{e}_z$, ($c \in \mathbb{C}$, $k \in \mathbb{R}$)

$$(c-U)(\phi'' - k^2\phi) + U''\phi = 0, \qquad \phi(-L) = \phi(L) = 0$$

(where $'$ is the $x$-derivative)
If there exists an eigenvalue $c$ with $\mathrm{Im}\, c > 0$, the flow is unstable.



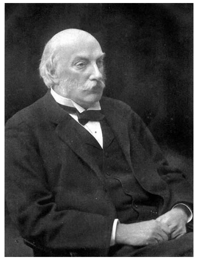
Kelvin-Helmholtz instability:



- One of the most classical hydrodynamic stability problem
- But, the stability condition on $U(x)$ is still nontrivial.

$$\boxed{U''(x) \neq 0 \text{ everywhere}} \; \substack{\Longrightarrow \\ \not\Longleftarrow} \; \boxed{\text{Stable}}$$ (Rayleigh 1880)

# History

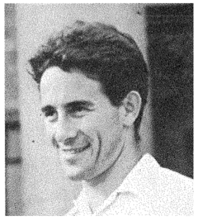


- 1880 Rayleigh
  No inflection point ($U'' \neq 0$) $\Rightarrow$ Stable

- 1950 Fjørtoft
  One inflection point $x_I$ and $U''(U-U_I)>0$ where $U_I=U(x_I)$ $\Rightarrow$ Stable

- 1964 Rosenbluth & Simon (Nyquist method)
  In the limit $k \to 0$,

$$\left.\frac{1}{U'(U-U_I)}\right|_{-L}^{L}+\int_{-L}^{L}\frac{U''}{U'^2(U-U_I)}dx>0 \Leftrightarrow \text{Stable}$$



- 1969 Arnold (variational method)
  $\delta^2 E$ is poitive or negative definite $\Rightarrow$ Stable
- 1991 Barston (variational method)
- 1999 Balmforth & Morrison (Nyquist method) a necessary and sufficient condition$\star$
- 2003, 2005 Lin (Tollmien's method) a necessary condition$\star$

$\star$ These methods requirs a solution of Rayleigh's equation.

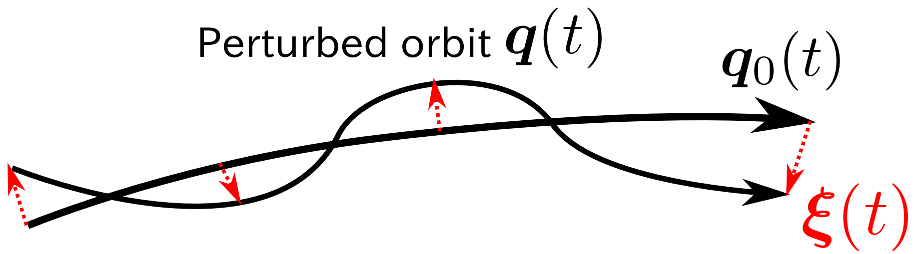
- 2014 Hirota, Morrison & Hattori (variational method)
  a necessary and sufficient stability condition

# General pertubation theory

Lagrangian: $L[\boldsymbol{q}] = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$

$$\boldsymbol{q}(t) = \boldsymbol{q}_0(t) + \boldsymbol{\xi}(t)$$

$\boldsymbol{q}_0$: known solution, $\boldsymbol{\xi}$: perturbation



Weakly nonlinear dynamics

Linearized dynamics

$$L = L_0 + \cancel{\delta L[\boldsymbol{\xi}]} + \boxed{\delta^2 L[\boldsymbol{\xi}, \boldsymbol{\xi}]} + \underline{\delta^3 L[\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}]} + \underline{\delta^4 L[\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}]} + \dots$$

0 at equilibrium

"Two-mode" coupling

$\omega_a = \omega_b$

- Linear instability
- Resonant absorption/growth

Three-mode coupling

$\omega_a = \omega_b + \omega_c$

- Parametric instability
- Second harmonic resonance etc.

Four-mode coupling

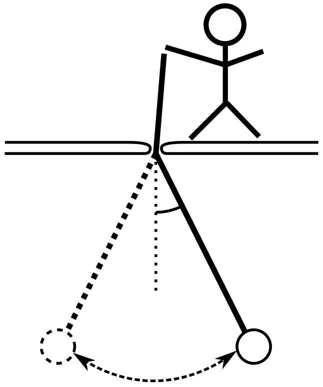
$\omega_a = \omega_b + \omega_c + \omega_d$

- Modulational instability
- Nonlinear frequency shift etc.

- Analysis of $\xi$ is, however, tedious when $\xi$ has a large degree of freedom.
- Each mode coupling can be reduced to a "normal form" by the transformation to action-angle variables.

# Action-angle variables in classical mechanics

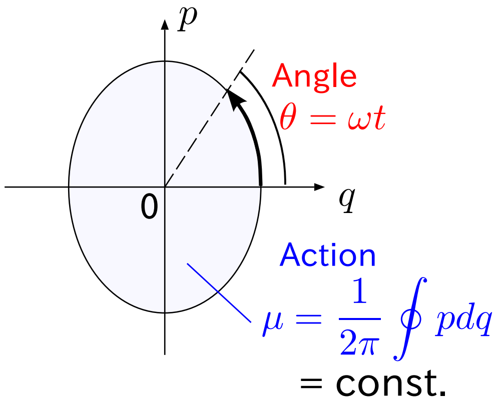
Periodic motion ··· $\boxed{\text{Canonical variables } (q,p)} \Rightarrow \boxed{\text{Action-angle variables } (\mu,\theta)}$



$$H = \omega\frac{q^2+p^2}{2} = \omega\mu$$

Normal form of single mode

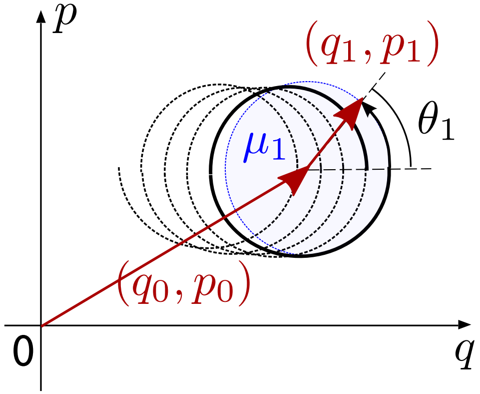
Energy = Frequency × Action



- Adiabatic invariance

  $\mu \simeq$ const. when a parameter (such as $\omega$) is slowly varying.

- Averaging



If $(q_1, p_1)$ is fast oscillation,

$$H \simeq H_0(q_0,p_0) + \omega_1(q_0,p_0)\mu_1$$

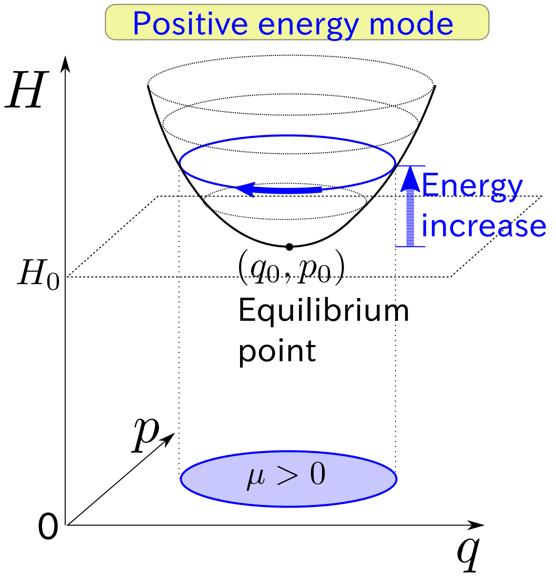
⇒ Averaged equation for $(q_0, p_0)$

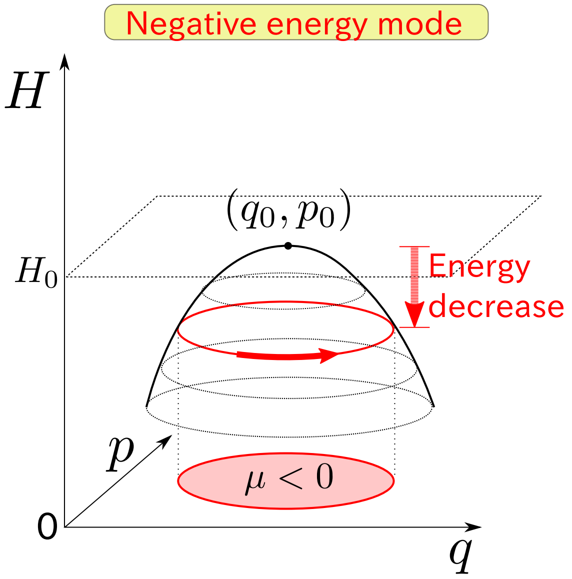
# • Instability

- ▷ An instability is caused by a resonance $\omega_1 = \omega_2$ between positive and negative energy modes. (Krein 1950)

$$H = H_0 + \omega_1 \mu_1 + \omega_2 \mu_2$$

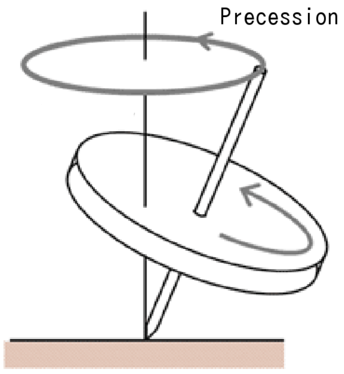
$$> 0 \qquad < 0$$





- ▷ Negative energy mode is also destabilized by energy dissipation effect.

Ex. Precession of spinning top



∴ $\boxed{\text{Signs of modal energies (called Krein signatures) are important!}}$

# Variational stability conditions

Linear Hamiltonian system with $N$ degrees of freedom
$u = (q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$;

$$\begin{cases} \partial_t \boldsymbol{q} = \partial H / \partial \boldsymbol{p}, \\ \partial_t \boldsymbol{p} = -\partial H / \partial \boldsymbol{q}, \end{cases} \quad \Leftrightarrow \quad \partial_t u = \mathcal{J}\mathcal{H}u, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad H = \frac{1}{2}\langle u, \mathcal{H}u \rangle$$

$$\Leftrightarrow \quad i\partial_t u = \mathcal{L}u, \quad \mathcal{L} = i\mathcal{J}\mathcal{H}$$

where $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{J}^* = -\mathcal{J}$. But, $\mathcal{L}$ is non-self-adjoint $\mathcal{L}^* \neq \mathcal{L}$.

**Lyapunov stability theorem** (Oberman and Kruskal 1965, Case 1965, Barston 1977)

$Q := \langle \overline{u}, i\mathcal{J}\mathrm{P}(\mathcal{L})u \rangle$ is a constant of motion, where $\mathrm{P}$ is any real polynomial.

$$\exists \mathrm{P} \ \text{ and } \ \exists \epsilon > 0 \ \text{ s.t. } \ Q \geq \epsilon \|u\|^2 \text{ or } -Q \geq \epsilon \|u\|^2 \quad \Rightarrow \quad \text{Stable}$$

(Hint) If $\mathrm{P}(\mathcal{L}) = \mathcal{L}/2$, then $Q = H$.
If $\mathrm{P}(\mathcal{L}) = \mathcal{L}^3$, $i\mathcal{J}\mathrm{P}(\mathcal{L}) = \mathcal{H}(i\mathcal{J})\mathcal{H}(i\mathcal{J})\mathcal{H}$ is also self-adjoint.

Lots of sufficient stability conditions$\cdots$ What choice of $\mathrm{P}$ leads to a better condition?

Modal decomposition:

$$u = \sum_{\alpha=1}^{2N} u_\alpha e^{-i\omega_\alpha t} = \sum_{\alpha=1}^{2N} u_{\overline{\alpha}} e^{-i\overline{\omega_\alpha} t}$$

Eigenvalue problem: $\mathcal{E}(\omega_\alpha)u_\alpha = \mathcal{E}(\overline{\omega_\alpha})u_{\overline{\alpha}} = 0$ where $\mathcal{E}(\omega) = \omega i\mathcal{J} - \mathcal{H}$.

Accordingly, $Q$ is decomposed into

$$Q = \sum_{\alpha=1}^{2N} \mathrm{P}(\omega_\alpha)\mu_\alpha \quad \text{with} \quad \mu_\alpha = \langle \overline{u_{\overline{\alpha}}}, i\mathcal{J}u_\alpha \rangle = \overline{q_{\overline{\alpha}1}} \frac{1}{\mathcal{E}_{1,1}(\omega_\alpha)} \frac{\partial D}{\partial \omega}(\omega_\alpha) q_{\alpha 1}$$

where $\mu_\alpha$ corresponds to the action variable for a neutrally stable mode ($\omega_\alpha \in \mathbb{R}$).

- $\mathcal{E}_{1,1}$ is the $(1,1)$ cofactor of the matrix $\mathcal{E}$.
- $D(\omega) = \det|\mathcal{E}(\omega)|$ is the characteristic polynomial; $D(\omega_\alpha) = 0$.

Even when $H = \frac{1}{2}\sum_{\alpha=1}^{2N} \omega_\alpha \mu_\alpha$ is indefinite, it is possible to make $Q$ positive or negative definite by choosing $\mathrm{P}$.

Theorem: If a polynomial $\mathrm{P}(\omega)$ is chosen such that $\frac{\mathrm{P}(\omega)}{\mathcal{E}_{1,1}(\omega)}\frac{\partial D}{\partial\omega}(\omega) \le 0$ holds for all $\omega \in \mathbb{R}$, then

$$\max_u \frac{Q}{|u|^2} > 0 \quad \Leftrightarrow \quad \text{Spectrally unstable; } \mathrm{Im}\,\exists\omega_\alpha > 0$$

(Necessary and sufficient condition)

$\because$ $Q \le 0$ for all neutrally stable modes
$Q \gtrless 0$ only for growing ($\mathrm{Im}\,\omega_\alpha > 0$) and damping ($\mathrm{Im}\,\overline{\omega_\alpha} < 0$) modes

- A trivial choice is $\mathrm{P}(\omega) = -\mathcal{E}_{1,1}(\omega)\frac{\partial D}{\partial\omega}(\omega)$. But, this choice is not practically useful because $D(\omega)$ is needed to construct $Q$.
- What is interesting here is that

$$\underline{\text{Number of unstable eigenvalues } (\mathrm{Im}\,\omega_\alpha > 0) \text{ of the non-self-adjoint } \mathcal{L}}$$

$$\Updownarrow \text{ one to one relation}$$

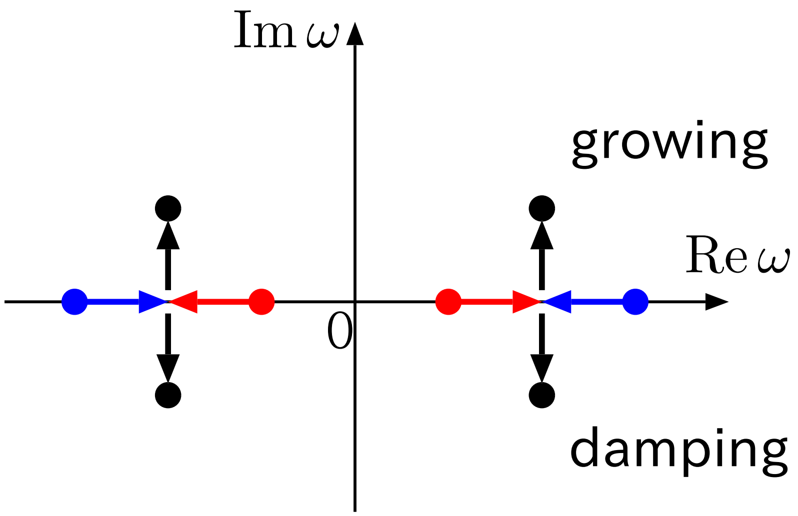
$$\underline{\text{Number of positive eigenvalues of the self-adjoint } \mathcal{Q}} \text{ where } Q = \langle \overline{u}, \mathcal{Q}u \rangle$$

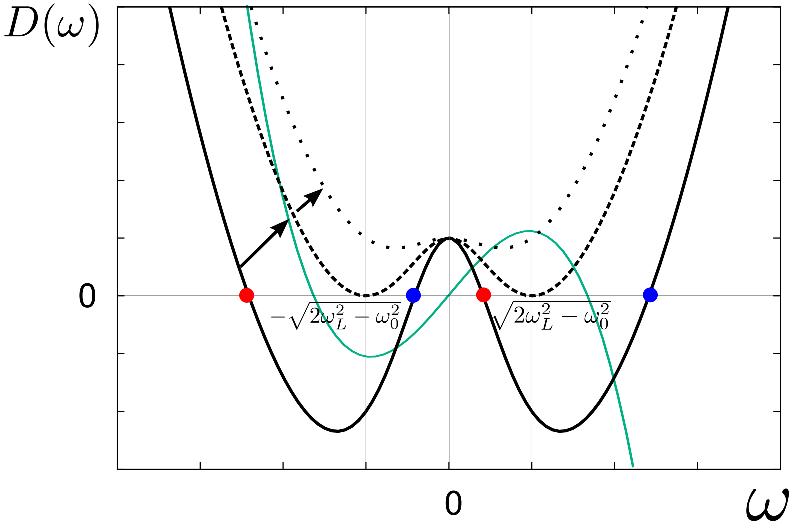
# Example

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}q_1\\ q_2\end{pmatrix} + 2\begin{pmatrix}0 & -\omega_L\\ \omega_L & 0\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \omega_0^2\begin{pmatrix}q_1\\ q_2\end{pmatrix}$$

Coriolis force Potential force

An instability is caused by resonance between a positive energy mode and a negative energy mode.

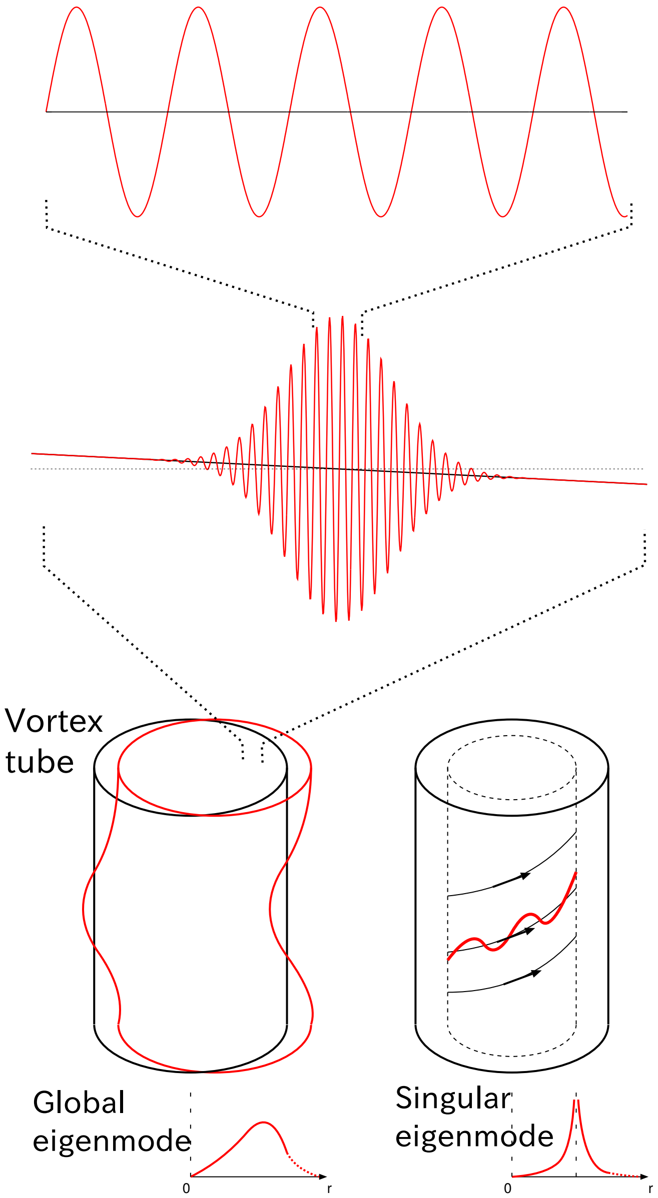




$$D(\omega) = \omega^4 - 2(2\omega_L^2 - \omega_0^2)\omega^2 + \omega_0^4$$
$$\mathcal{E}_{1,1}(\omega) = \omega^2 + \omega_0^2$$

By choosing $\mathrm{P}(\omega) = -(\omega^2+\omega_0^2)\omega(\omega^2-2\omega_L^2+\omega_0^2)$, we obtain a necessary and sufficient condition as $\max Q/\|u\|^2 > 0 \Leftrightarrow$ Unstable.

# Wave action theory for fluid (mode ⇒ wave)



### Uniform background

- Plane wave: $A\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)$
- Dispersion relation: $D(\omega,\boldsymbol{k})=0$
- Wave action $=\dfrac{\partial D}{\partial\omega}(\omega,\boldsymbol{k})|A|^2$ (Auer *et al.* 1958)

### Weakly non-uniform background (short wavelength limit)

- Wave packet: $A(\boldsymbol{x})\exp(i\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)$,
- Local dispersion relation: $D(\omega,\boldsymbol{k},\boldsymbol{x})=0$
- Wave action density $=\dfrac{\partial D}{\partial\omega}(\omega,\boldsymbol{k},\boldsymbol{x})|A(\boldsymbol{x})|^2$

⇒ Wave-kinetic theory & Weak turbulence theory
(Stix 1962, Sagdeev & Galeev 1969)

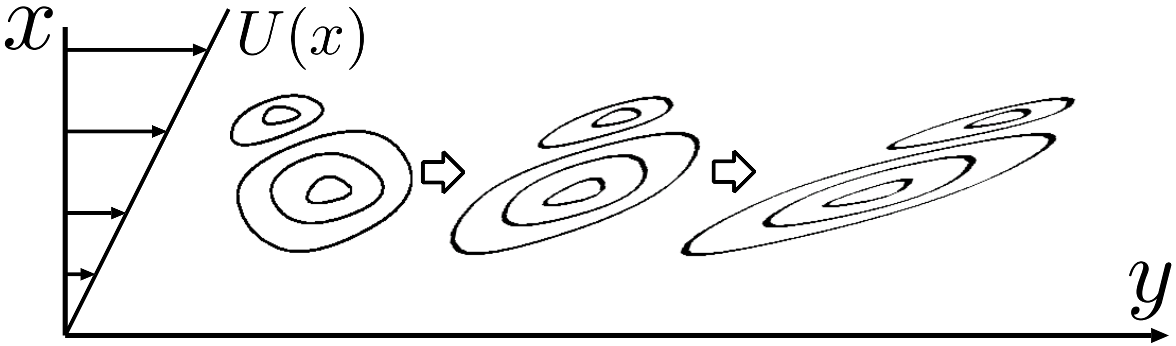
### Non-uniform background

Eigenvalue problem ⇒ discrete and continuous spectra
(Differential equation)

Wave action (= action variable) is nontrivial.

# Continuous spectrum in hydrodynamic disturbance

Vorticity disturbance stretched by background shear flow $U(x)$ (Case 1960)



- Initial condition $e^{iky}$ $\Rightarrow$ $e^{iky-ikU(x)t}$ : the Doppler shift $kU(x)$ depends on $x$.
  $\Rightarrow$ Continuous spectrum $\{kU(x)|x \in \mathbb{R}\}$
- “Continuum mode”
  Integral of infinite number of singular eigenmodes localized on each streamline
  e.g. delta functions

Since the singular function is not square integrable, it has been difficult to calculate wave action (and also wave energy) for continuous spectrum.

· Van Kampen mode (Morrison & Pfirsch 1992),
· Rayleigh equation (Balmforth & Morrison 2002)
· General singular mode (Hirota & Fukumoto 2008)

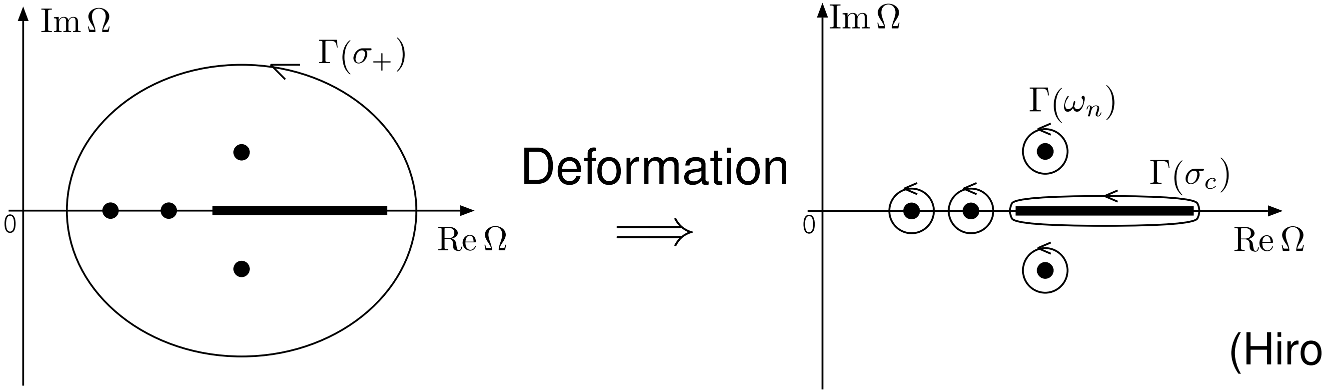
# Action variables for eigenmode and continuum mode

$$\boxed{i\partial_t u = \mathcal{L}u,\ \mathcal{L}\mathcal{J} = \mathcal{J}\mathcal{L}^*}$$

Laplace transform $u(t) \mapsto \mathsf{U}(\Omega) = \frac{iu(0)}{\Omega - \mathcal{L}}$, and define $D^{-1}(\Omega) := \left\langle \overline{u(0)}, i\mathcal{J}\mathsf{U}(\Omega) \right\rangle$

$$S = \frac{1}{4\pi}\int_0^{2\pi} \left\langle u, \mathcal{J}^{-1}\frac{\partial u}{\partial \theta} \right\rangle d\theta = \frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)} D^{-1}(\Omega) d\Omega = \sum_n \mu_n + \int_{\sigma_c} \mu(\omega) d\omega$$

- Eigenvalues $\{\omega_n | n = 1, 2, \ldots\}$, $\quad \mu_n = \frac{1}{2\pi i}\oint_{\Gamma(\omega_n)} D^{-1}(\Omega) d\Omega = \left(\frac{\partial D}{\partial \Omega}\right)^{-1}(\omega_n)$, (residue)
- Continuous spectrum $\omega \in \sigma_c \subset \mathbb{R}$ , $\quad \mu(\omega) = \frac{i}{2\pi}\left[D^{-1}(\omega + i0) - D^{-1}(\omega - i0)\right]$. (jump)



(Hirota & Fukumoto 2008)

# Rayleigh equation ~ Stability of inviscid parallel shear flow ~

Basic flow $\boldsymbol{U} = U(x)\boldsymbol{e}_y$, Disturbance $\tilde{\boldsymbol{u}} = \nabla[\phi(x)e^{-i\omega t+iky} + \text{c.c.}] \times \boldsymbol{e}_z$, ($\omega \in \mathbb{C}$, $k \in \mathbb{R}$)

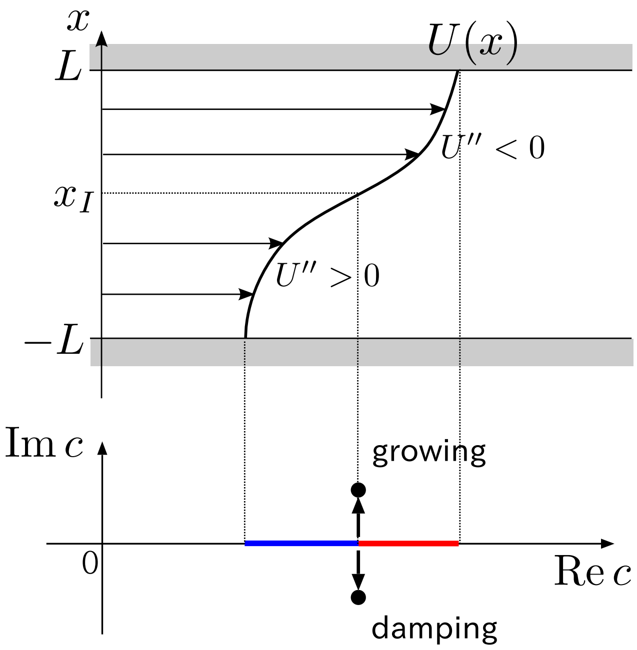
$$(c - U)(\phi'' - k^2\phi) + U''\phi = 0, \qquad \phi(-L) = \phi(L) = 0$$

If there exists an eigenvalue $c = \omega/k$ with $\mathrm{Im}\, c > 0$, the flow is spectrally unstable.

- (Case 1960) A continuous spectrum exists,
  $c = \omega/k \in \{U(x) \in \mathbb{R} \mid x \in [-L, L]\}$.
- Sign of the energy of continuous spectrum
  $=$ Sign of $UU''$
  (Balmforth & Morrison 2002, Hirota & Fukumoto 2008)
- Kelvin-Helmholtz instability emerges from a contact point between positive- and negative-energy continuous spectra.

$\cdots$ Analogous to Krein's theory

(Hagstrom & Morrison 2011)



# Variational stability conditions

In terms of vorticity disturbance $w = -\Delta\phi := -\phi'' + k^2\phi$,

$$\frac{i}{k}\partial_t w = \mathcal{L}w \quad \text{where} \quad \mathcal{L} = U - U''\Delta^{-1}$$

$\mathcal{L}$ is non-self-adjoint ($\mathcal{L} \neq \mathcal{L}^*$), but has a Hamiltonian property $\mathcal{L}U'' = U''\mathcal{L}^*$.

**Theorem** [Oberman and Kruskal 1965, Case 1965, Barston 1977]

Let $\mathrm{P}(c)$ be any real polynomial. Then,

$$Q = \int_{-L}^{L} \overline{w}\frac{1}{U''}\mathrm{P}(\mathcal{L})w dx = \text{const.}$$

Therefore,

If $\exists\mathrm{P}$ and $\epsilon > 0$ s.t. $Q \geq \epsilon\|u\|^2$ or $-Q \geq \epsilon\|u\|^2$ $\Rightarrow$ (Lyapunov) Stable

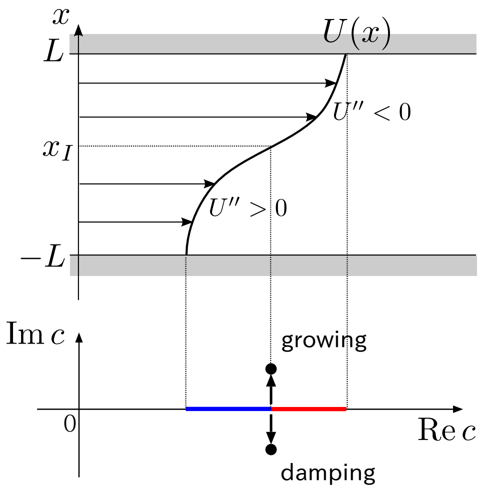
⇒ What is the best choice of $\mathrm{P}$?

- If $U(x)$ has only one inflection point $x = x_I$,

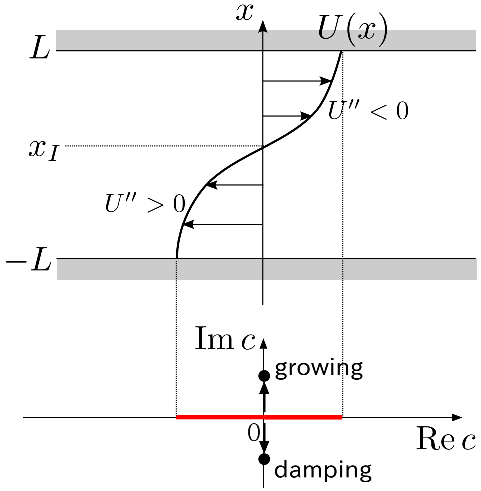
  the choice of $\mathrm{P}(c) = c - U(x_I)$ results in

> (Arnold 1966) The second variation of the energy in the inertial frame moving at the velocity $U_I = U(x_I)$ is
>
> $$Q = \delta^2 E_I = \int_{-L}^{L} \overline{w} \left( \frac{U - U_I}{U''} - \Delta^{-1} \right) w dx.$$
>
> The shear flow $U$ is stable if $\delta^2 E_I$ is either positive or negative definite.

(This includes Rayleigh-Fjørtoft's stability criterion.)



$\Rightarrow$



In this frame, the energy of the continuous spectrum is negative everywhere.

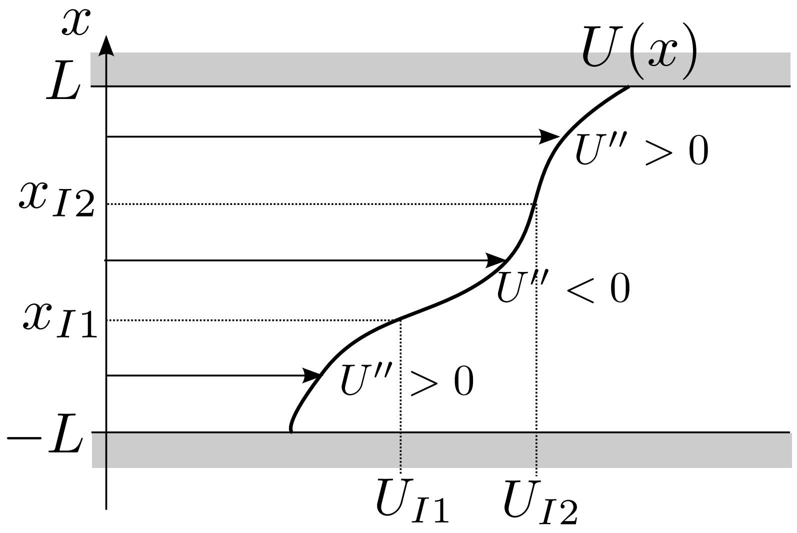
- If $U(x)$ has multiple inflection points $x_{In}$, $n = 1, 2, \ldots, N_I$,

the choice of $\mathrm{P}(c) = \prod_{n=1}^{N_I} (c - U_{In})$, where $U_{In} = U(x_{In})$, results in

(Barston 1991) The shear flow $U$ is stable if

$$Q = \int_{-L}^{L} w \frac{1}{U''} \prod_{n=1}^{N_I} \left[ (U - U_{In}) - U'' \Delta^{-1} \right] w dx,$$

is either positive or negative definite.



$\ldots$ still sufficient conditions, but very close to necessary and sufficient one.

Theorem: Assume

1) $U(x)$ is an analytic, bounded and strictly monotonic function on $[-L, L]$.

2) if $U''(x_I) = 0$ at $x = x_I$, then $U'''(x_I) \neq 0$.

Define the quadratic form $Q$ by choosing $\mathrm{P}(c) = \nu \prod_{n=1}^{N_I} (c - U_{In})$ where either $\nu = 1$ or $\nu = -1$ is chosen such that

$$\frac{\nu}{U''} \prod_{n=1}^{N_I} (U - U_{In}) \leq 0 \quad \text{for all } x.$$

Then,

$$\max \frac{Q}{\int |w|^2 dx} > 0 \quad \Leftrightarrow \quad \text{(Spectrally) Unstable}$$

(Necessary and sufficient stability condition!)

(Hirota, Morrison, Hattori 2014)

By showing $Q > 0$ for some $w$, we can prove instability!

# Outline of the proof:

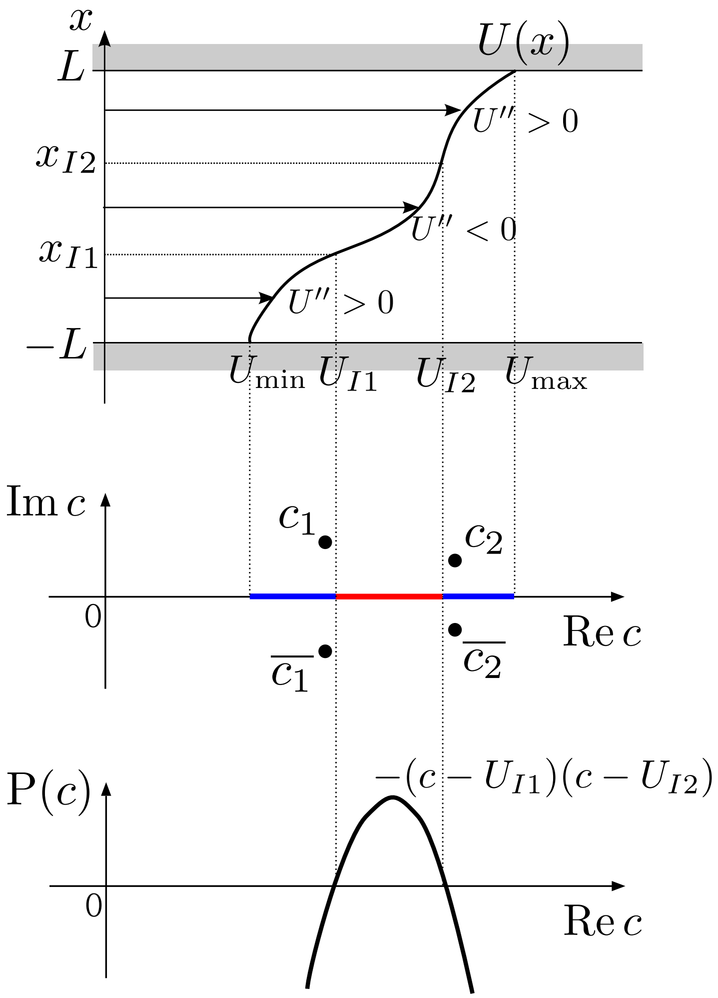
- Spectrum: $Sp(\mathcal{L}) = \{c_\alpha, \overline{c_\alpha} \in \mathbb{C}\,;\ \mathrm{Im}\, c_\alpha \neq 0, \alpha = 1, 2, \dots, N\} \cup \{U(x) \in \mathbb{R}\,;\ x \in [-L, L]\}$
  (discrete) (continuous)

- Mode decomposition:

$$w = \sum_{\alpha=1}^{N} \left( w_\alpha e^{-ikc_\alpha t} + w_{\overline{\alpha}} e^{-ik\overline{c_\alpha} t} \right) + \int_{U_{\min}}^{U_{\max}} \hat{w}(c) e^{-ikct} dc$$

Exponentially growing and damping modes

Neutrally stable continuum mode



- Then, $Q$ is also decomposed into

$$Q = \sum_{\alpha=1}^{N} \underbrace{[\mathrm{P}(c_\alpha)\mu_\alpha + \mathrm{P}(\overline{c_\alpha})\overline{\mu_\alpha}]}_{\gtrless 0} + \int_{U_{\min}}^{U_{\max}} \underbrace{\mathrm{P}(c)\hat{\mu}(c)}_{\leq 0} dc$$

where $\mu_\alpha = \int_{-L}^{L} \overline{w_{\overline{\alpha}}} w_\alpha / U'' dx$

and $\mathrm{sgn}\, \hat{\mu}(c) = \mathrm{sgn}\, U''(U^{-1}(c))$.

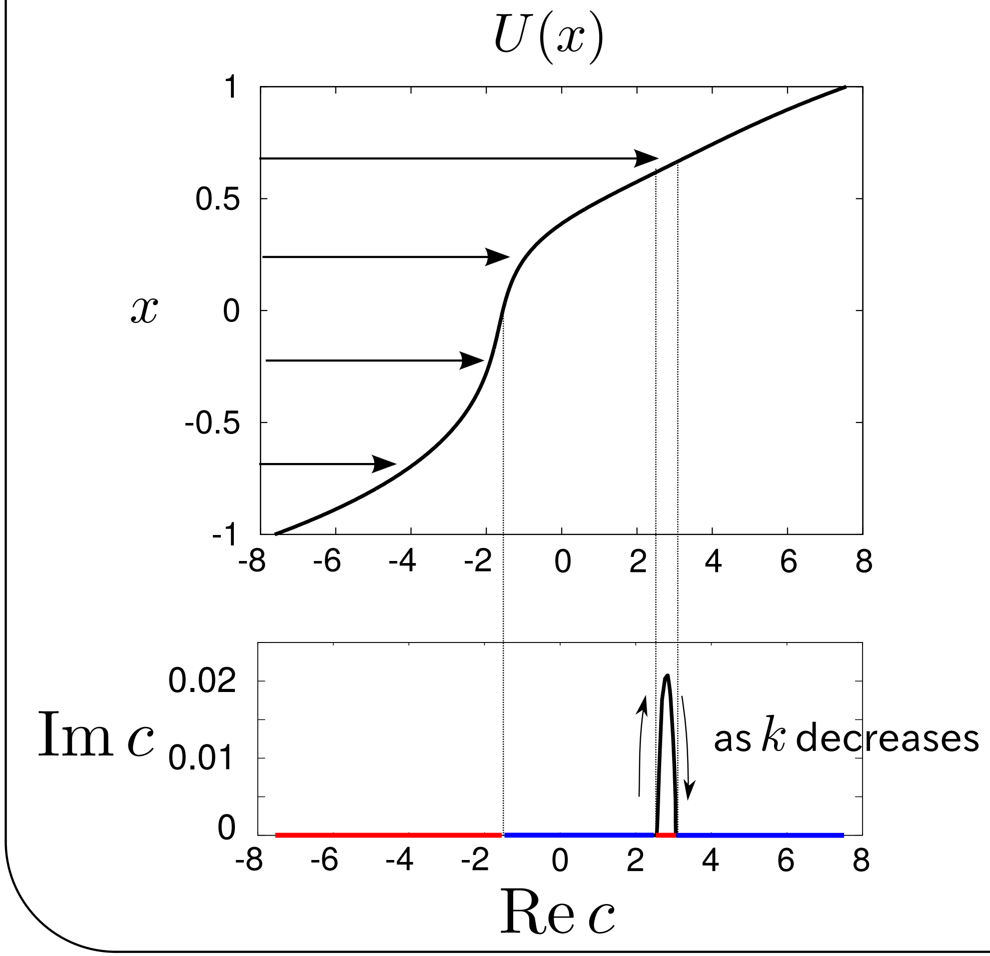
(Balmforth & Morrison 2002, Hirota & Fukumoto 2008)

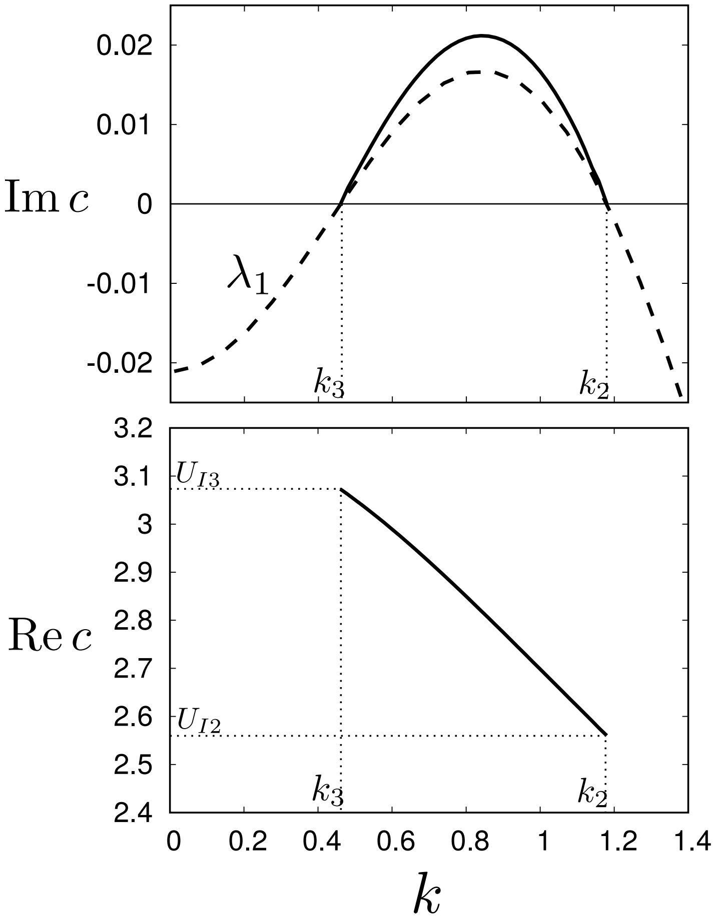
$\therefore$ If $Q > 0$ for some $w$, there exists at least a growing eigenmode. $\square$

# Numerical tests

$$\boxed{\lambda_1 = \max Q / \int |w|^2 dx}$$

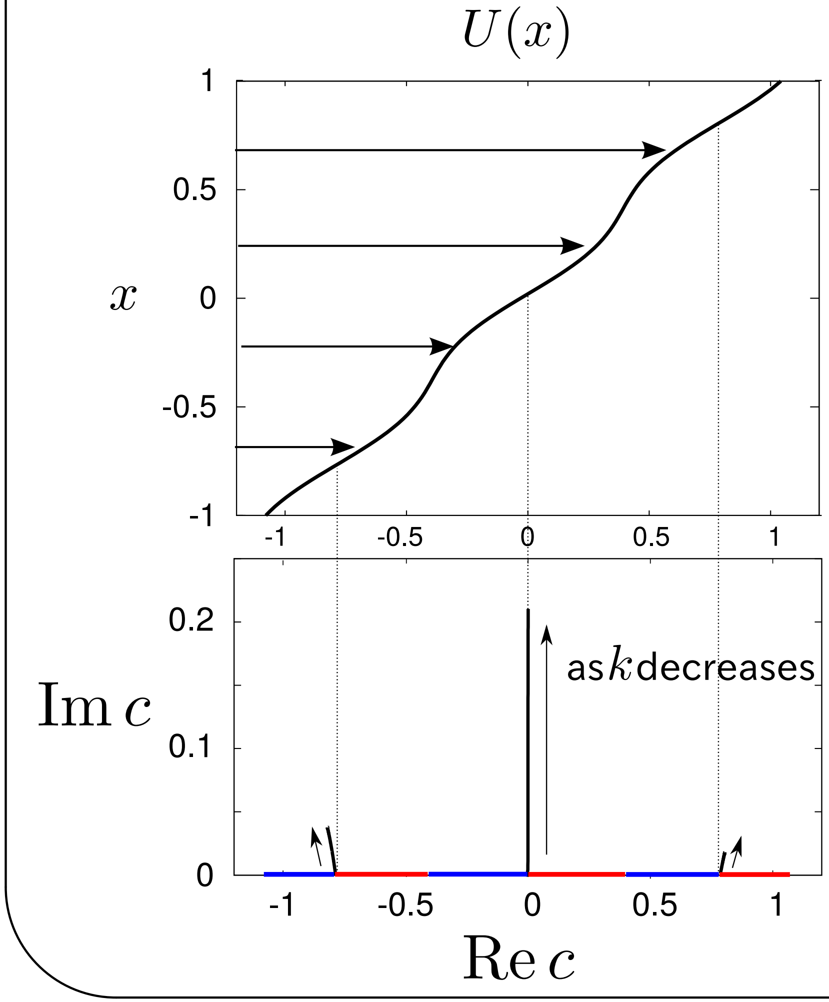
## Example 1 $U(x) = x + 5x^3 + 1.62 \tanh[4(x - 0.5)], \quad x \in [-1, 1]$

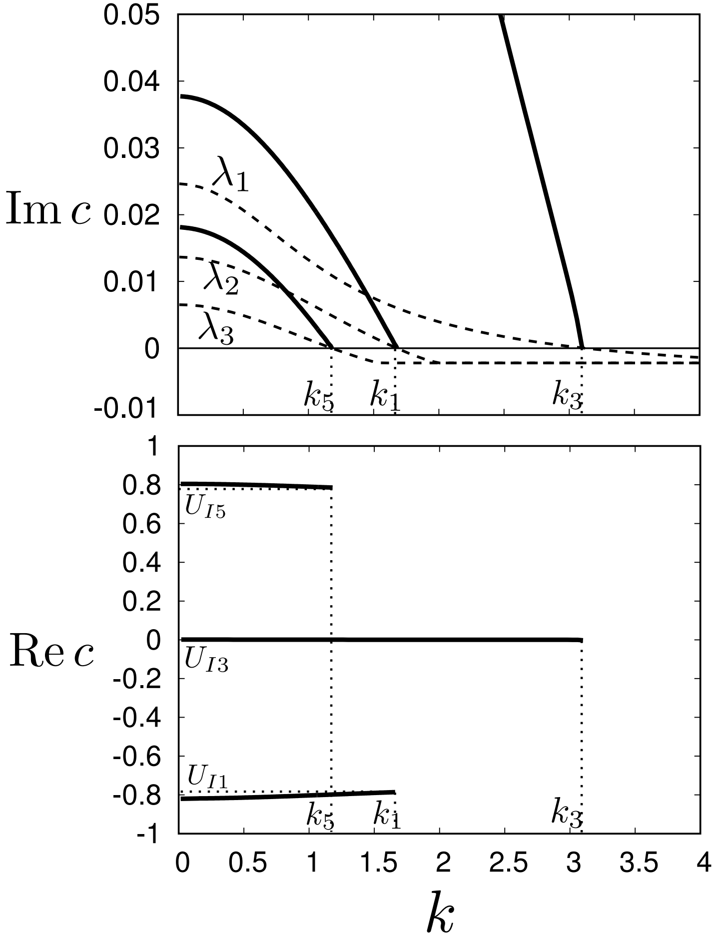




Corollary: At the marginal stability $\lambda_1 \simeq 0$, $Q$ is non-singular ($\int |w_1|^2 dx < \infty$).

## Example 2 $U(x) = x - 0.02 + \sin[8(x - 0.02)]/16, \quad x \in [-1, 1]$





Corollary: Number of positive signatures of $Q$ corresponds to number of unstable eigenmodes.

# 5. Summary

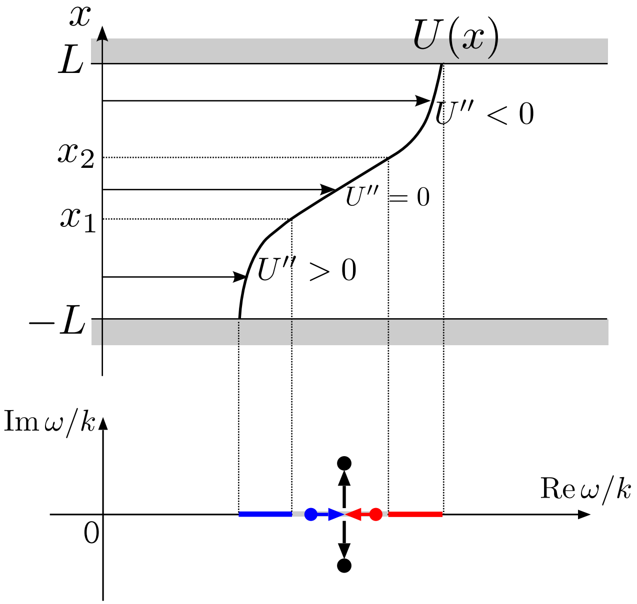
- Using Rayleigh's equation, we have demonstrated that the variational method can be improved to give necessary and sufficient stability criteria.

| Linearized system $i\partial_t w = k\mathcal{L}w$ | Variational criterion $\max Q/\|w\|^2 > 0$ |
|---|---|
| non-self-adjoint | self-adjoint |
| $c_1, c_2, \cdots \in \mathbb{C}$ | $\lambda_1 > \lambda_2 > \cdots \in \mathbb{R}$ |
| $w(x) \in \mathbb{C}$ | $w(x) \in \mathbb{R}$ |
| $\exists j, \operatorname{Im} c_j > 0 \Leftrightarrow$ Unstable | $\lambda_1 > 0 \Leftrightarrow$ Unstable |
| singular as $\operatorname{Im} c \to +0$ | non-singular around $\lambda = 0$ |

- We can determine the stability more efficiently and accurately than directly solving the linearized equation.

However, I have a lot of open questions.

- This variational method can calculate neither growth rates nor frequencies of the unstable modes.
- So far, this method is not useful for systems of finite degree of freedom.
- Piecewise-linear flows are more complicated.

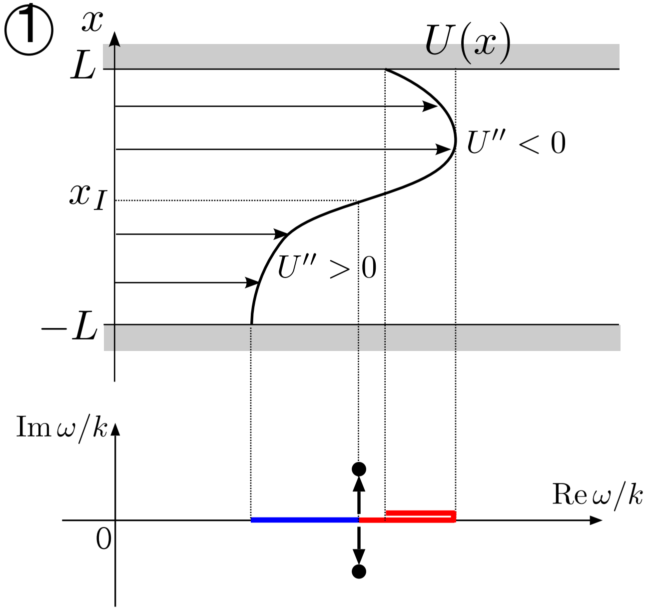


– The linear part $[x_1, x_2]$ is a set of inflection points, on which the classical Krein collision occurs.

$$\min_{x_I \in [x_1, x_2]} \max_{\|w\|=1} Q|_{\mathrm{P}=c-U(x_I)} > 0 \quad \Leftrightarrow \quad \text{Ustable}$$

- The variational method is not always applicable to nonmonotonic shear flows,

①



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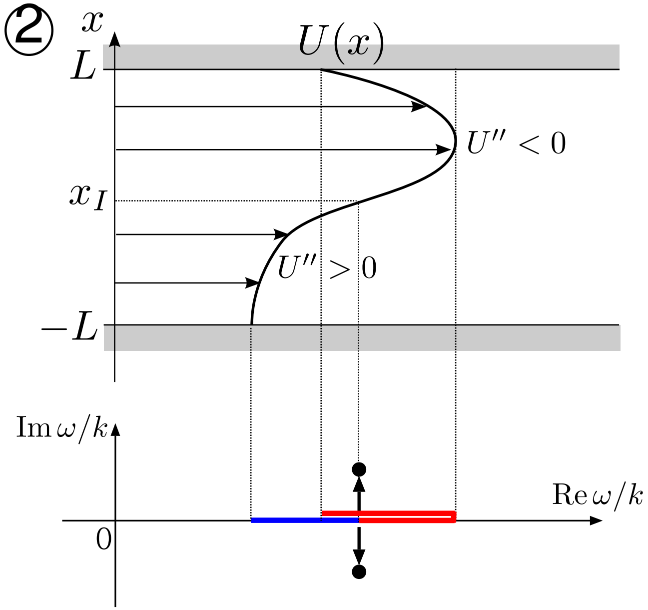

Impossible to make all signatures negative

- We can think of many difficult situations when multiple continuous spectra exist. (e.g., stratified shear flow, MHD etc.)
- Smoothness of shear flow is mathematically important, but physically not.
- Is it possible to obtain an unified view of resonance among discrete modes and continuum modes? (like the normal forms)

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