Mathematical Analysis of various Geophysical Flows

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Geophysical Flows

Outline of the program

- General Conservation Laws and 2nd-Law of Thermodynamics
  - Conservation of mass, momentum, energy
  - Entropy
- Physical understanding of oceanic and atmospheric flows
  - geostrophic approximation and hydrostratic law
  - Coriolis force
  - balance gravitation and rotation : the Taylor-Proudham Theorem
  - shallow water models
  - stratified fluids
  - friction forces and Ekman boundary layers
- Mathematical Equations
  - Navier-Stokes-Coriolis equations
  - Primitive equations
  - Ekman layers
  - Stratified heat-conducting fluids
  - Quasigeostrophic equations
- Key Mathematical Results and Open Problems
  - unique, global, strong solvability for large data with/without fast rotation, periodic solutions for small/large forces, stability issues

Balance Laws for Mass, Momentum and Energy

The balance laws for mass, momentum and energy read as

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0 && \text{in } \Omega, \\ \rho(\partial_t + u\cdot\nabla)u + \nabla\pi &= \operatorname{div} S && \text{in } \Omega, \\ \rho(\partial_t + u\cdot\nabla)\epsilon + \operatorname{div} q &= S : \nabla u - \pi \operatorname{div} u && \text{in } \Omega, \\ u = 0, \quad q\cdot\nu &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- ρ density, u velocity, π pressure, ϵ internal energy, S extra stress and q heat flux.
- This gives conservation of the total energy since

$$\rho(\partial_t + u\cdot\nabla)e + \operatorname{div}(q + \pi u - Su) = 0 \quad \text{in } \Omega,$$

  with $e := |u|^2/2 + \epsilon$ energy density (kinetic and internal).
- Integrating over Ω yields

$$\partial_t \mathsf{E}(t) = 0, \quad \mathsf{E}(t) = \mathsf{E}_{kin}(t) + \mathsf{E}_{int}(t) = \int_\Omega \rho(t,x)e(t,x)dx,$$

  provided $q\cdot\nu = u = 0$ on $\partial\Omega$

Basic Laws from Thermodynamics

- Ansatz : free energy $\psi = \psi(\rho, \theta)$.
- Then

$$\epsilon = \psi + \theta\eta \quad \text{internal energy},$$

$$\eta = -\partial_\theta \psi \quad \text{entropy},$$

$$\kappa = \partial_\theta \epsilon = -\theta \partial_\theta^2 \psi \quad \text{heat capacity}.$$

- classical case, Clausius-Duhem equation reads as

$$\rho(\partial_t + u\cdot\nabla)\eta + \mathrm{div}(q/\theta) = S : \nabla u/\theta - q\cdot\nabla\theta/\theta^2 + (\rho^2\partial_\rho - \pi)(\mathrm{div}\, u)/\theta \quad \text{in } \Omega.$$

- Hence, entropy flux Φ_η is given by $\Phi_\eta := q/\theta$
- entropy production by

$$\theta r := S : \nabla u - q \cdot \nabla\theta/\theta + (\rho^2 \partial_\rho - \pi)(\mathrm{div}\, u)$$

- boundary conditions employed yield that for total entropy N we have

$$\partial_t \mathrm{N}(t) = \int_\Omega r(t,x)dx \geq 0, \quad \mathrm{N}(t) = \int_\Omega \rho(t,x)\eta(t,x)dx,$$

  provided $r \geq 0$ in Ω.
- $\mathrm{div}\, u$ has no sign, hence $\pi = \rho^2 \partial_\rho \psi$, Maxwell's relation.
- this leads to $S : \nabla u \geq 0$ and $q \cdot \nabla\theta \leq 0$.

Summary

- Summarizing : conservation of energy and total entropy is non-decreasing provided these conditions, Maxwell and (BC) are satisfied, independent of special form of stress S and heat flux q
- Thus, these conditions ensure thermodynamical consistency of the model.
- example : classical laws due to Newton and Fourier :

$$S := S_N := 2\mu_s D + \mu_b \mathrm{div}\, u\, I, \quad 2D = (\nabla u + [\nabla u]^{\mathsf{T}}), \quad q = -\alpha_0 \nabla \theta.$$

- thermodynamically consistent if $\mu_s \geq 0$, $2\mu_s + n\mu_b \geq 0$ and $\alpha_0 \geq 0$

Some Physics for Oceanic and Atmosheric Flows

- thin spherical layer of fluid
- Geostrophic and Hydrostatic Approximation
  - first dominating force is gravity : hydrostatic law

$$\varrho = \varrho(x_3), \quad p = p_0(x_3) \text{ with } (p_0)_{x_3} = -\varrho g$$

  - aspect ratio ; $\delta = \frac{D}{L}$
  - Coriolis force : time for fluid with speed u to cross distance L is L/U
  - if this time is small compared to period of rotation $|\Omega|^{-1}$, fluid does not feel rotation
  - rotation important only if Rossby number $\varepsilon = \frac{U}{2|\Omega|L}$ is small, realistic range $\varepsilon \sim 0.05$
  - balance between gravity and rotation :

$$(\Omega \cdot \nabla)u - \Omega \nabla \cdot u = -\frac{\nabla \varrho \times \nabla p}{2\varrho^2}$$

  if δ small, then only vertical component of rotation $f = |\Omega| \sin \theta$ is dynamically significant. Hence :

$$(fe_3 \cdot \nabla)u_h = -\frac{(\nabla \varrho \times \nabla p)_h}{2\varrho^2}$$

$$fe_3 \cdot u_h = 0$$

The Taylor-Proudman Theorem and Approximations

- if fluid is barotropic, then $(fe_3 \cdot \nabla)u_h = 0$
- if fluid is incompressible, then $(fe_3 \cdot \nabla)u_3 = 0$
- this means all 3 components of velocity are independent of x_3. This is the Taylor-Proudman theorem
- specify to Coriolis force : $2\varrho\Omega \times u = -\nabla p - \varrho g e_3$
- δ small implies that only fe_3 is dynamically significant. This yields

$$u_h = \frac{1}{f\,\varrho_0} e_3 \times \nabla p \qquad \text{geostrophic approximation}$$

$$\varrho g = -\partial_3 p \qquad \text{hydrostatic approximation}$$

- geostrophic approximation : balance between $\nabla_H p$ and horizontol component of Coriolis force
- hydrostatic approximation : balance between $\partial_3 p$ and gravity

Departures from Geostrophy : Waves in Shallow Water

- shallow layer of incompressible and inviscid fluid : fluid described by height H : fluctuation η around refence height H_0, purely horizontal velocity u
- δ small yields $u \cdot \nabla(f/H_0) = 0$
- pertubations of (η, u) yield

$$(\eta_{tt} + f^2\eta + \nabla \cdot (c_0^2 \nabla \eta))_t - gf((H_0)_x \eta_y - (H_0)_y \eta_x) = 0$$
$$(u_1)_{tt} + f^2 u_1 = -g(\eta_{xt} + f\eta_y)$$
$$(u_2)_{tt} + f^2 u_2 = -g(\eta_{yt} - f\eta_x)$$

  with shallow water speed $c_0 = (gH_0)^{1/2}$.
- consider solutions of form $\exp(i[\sigma t + k_1 x_1 + k_2 x_2])$
- gravity waves : Poincaré waves : $\sigma^2 = f^2 + c_0^2 k^2$
- Kelvin waves : $\sigma^2 = c_0^2 k_1^2$
- planetary waves : Rossby waves : ...

Effects of Stratification

- recall hydrostatic equilibrium :
  $\varrho = \varrho(x_3), \quad p = p_0(x_3)$ with $(p_0)_{x_3} = -\varrho g$
- $\partial_{tt} u_3 + N^2 u_3 = \varrho^{-1} \partial_{t3} p'$ with
- $N^2 = -g \varrho^{-1} \partial_3 \varrho_0$ buoyancy frequency

Dissipation from viscosity

- how to represent frictional forces $\mathcal{F}$?
- $\mathcal{F}$ proportional to ∇S, S stress tensor, coefficient is viscosity ν
- $\frac{\mathcal{F}}{\varrho} \sim \frac{\nu U}{L^2}$
- Ekman number E : ratio between frictional force per unit mass to Coriolis acceleration
- $E = \frac{\nu U / L^2}{1 \Omega U} = \frac{\nu}{2 \Omega L^2}$

Influence of Boundary Conditions

- so far : rotational effects studied in absence of boundaries
- example : stress induced by wind on ocean surface induces so-called Ekman transport
- Ekman flow causes mass to flow horizontally into some region and out of others
- This results vertical motion, e.g. vertical motion away from boundary in order to conserve mass
- vertical velocity produced is called Ekman pumping : this velocity distorts density field of ocean and causes wind-driven currents
- also bottom friction

Equation I : Equations of Navier-Stokes with Coriolis force

Simplifications : fluid incompressible, isothermal, no hor./vertical scaling but rotational effects

- I observer in non-rotating inertial frame. Then :

$$\left(\frac{dr}{dt}\right)_I = \left(\frac{dr}{dt}\right)_R + \Omega \times r$$

- Thus $u_I = u_R + \Omega \times r$
- Newton : Forces equal acceleration in inertial frame, thus

$$\begin{aligned}\left(\frac{du_I}{dt}\right)_I &= \left(\frac{du_I}{dt}\right)_R + \Omega \times u_I \\ &= \left(\frac{du_R}{dt}\right)_R + 2\Omega \times u_R + \Omega \times (\Omega \times r) + \frac{d\Omega}{dt} \times r\end{aligned}$$

- write centrifugal force as gradient : $\Omega \times (\Omega \times r) = -\nabla \frac{|\Omega \times r|^2}{2}$

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + 2\Omega \times u + \nabla p &= f, \quad \text{in } [0,T] \times \Omega \\ \operatorname{div} u &= 0, \quad \text{in } [0,T] \times \Omega \\ u &= 0, \quad \text{in } [0,T] \times \partial\Omega \\ u(0) &= u_0, \quad \text{in } \Omega \end{aligned}$$

Equation II : Hydrostatic Approx. : Primitive Equations

Primitive equations are fundamental model in geophysical flows, introduced by Lions, Temam and Wang in 1992-1993

Regularity results for the PEs 293

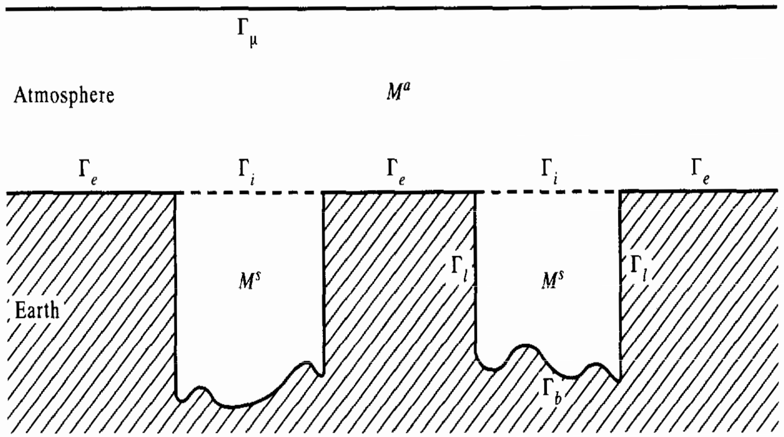


Fig. 1.

The domains M^a and M^s are open submanifolds of $S^2 \times \mathbb{R}$. We denote by M any one of the domains M^a or M^s. The Riemannian geometry of M is the same as that of $S^2 \times \mathbb{R}$ or $S^2 \times (0, 1)$. The tangent space $T_{(q,\xi)}M$ of M at $(q, \xi) \in M$ can be decomposed into the product of $T_q S^2$ and $\mathbb{R}$ as follows

$$T_{(q,\xi)}M = T_q S^2 \times \mathbb{R}. \qquad (1.6)$$

- $M_a = \mathbb{S}^2 \times (0,1)$, $M_s = \bigcup_{(\theta,\varphi)\in\Gamma_i} \{(\theta,\varphi) \times (-h(\theta,\varphi), 0)\}$
- both are submanifolds of $\mathbb{S}^2 \times \mathbb{R}$
- Riemannian metric g_M on M :

$$g_M((q,\xi),(v_1,w_1),(v_2,w_2)) = g_{\mathbb{S}^2}(q, v_1, v_2) + w_1 w_2, \quad v_1, v_2 \in T^2\mathbb{S}, w_1, w_2 \in \mathbb{R}$$

Primitive Equations on this manifold

- scaling argument taking into account different horizontal and vertical dimensions yields

$$\begin{cases} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi & = f, \qquad \text{in } M \times (0,T), \\ \text{div } u & = 0, \qquad \text{in } M \times (0,T), \\ \partial_t \tau + u \cdot \nabla \tau - \Delta \tau & = g_\tau, \qquad \text{in } M \times (0,T), \\ \partial_z \pi + 1 - \beta_\tau (\tau - 1) & = 0, \qquad \text{in } M \times (0,T), \end{cases}$$

- velocity $u = (v, w)$, where $v = (v_1, v_2)$ denotes the horizontal component and w the vertical one
- simplifying : consider $\Omega = G \times (-h, 0)$ where $G = (0,1) \times (0,1)$
- Boundary conditions :

$$\begin{cases} \partial_z v = 0, \quad w = 0, \quad \partial_z \tau + \alpha \tau = 0, & \text{on } \Gamma_u \times (0,\infty), \\ v = 0, \quad w = 0, \quad \partial_z \tau = 0, & \text{on } \Gamma_b \times (0,\infty), \\ v, \pi, \tau, \sigma \quad \text{are periodic} & \text{on } \Gamma_l \times (0,\infty), \end{cases}$$

where

$$\Gamma_u = G \times \{0\}, \quad \Gamma_b = G \times \{-h\} \quad \text{and} \quad \Gamma_l = \partial G \times (-h, 0),$$

and $\alpha > 0$.

Isothermal Situation

In isothermal situation, primitive equations are given by

$$\begin{aligned} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi &= f \quad \text{in } \Omega \times (0,T), \\ \partial_z \pi &= 0 \quad \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T), \\ v(0) &= a. \end{aligned} \tag{1}$$

- $\Omega = G \times (-h, 0)$, where $G = (0,1)^2$, $h > 0$

System is complemented by the set of boundary conditions

$$\begin{aligned} \partial_z v &= 0, \quad w = 0 && \text{on } \Gamma_u \times (0,T), \\ v &= 0, \quad w = 0 && \text{on } \Gamma_b \times (0,T), \\ & u,\ \pi \text{ are periodic} && \text{on } \Gamma_l \times (0,T). \end{aligned} \tag{2}$$

- $\Gamma_u := G \times \{0\}$, $\Gamma_b := G \times \{-h\}$, $\Gamma_l := \partial G \times [-h, 0]$

Equation III : Stratified Flows

Thermal disturbance about mean state in hydrostatic balance :

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \nu\Delta v + \nabla\pi - \Omega e_3 \times v &= \theta e_3 + g, \qquad \text{in } \mathbb{R}^3 \times (0,T), \\ \partial_t \theta + (v\cdot\nabla)\theta - \kappa\Delta v &= N^2 v_3 + h, \qquad \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} v &= 0, \qquad \text{in } \mathbb{R}^3 \times (0,T). \end{cases}$$

- N bouyancy frequency
- assume $\mu := \frac{\Omega}{N}$ fixed

Equation IV : Ekman Boundary Layers

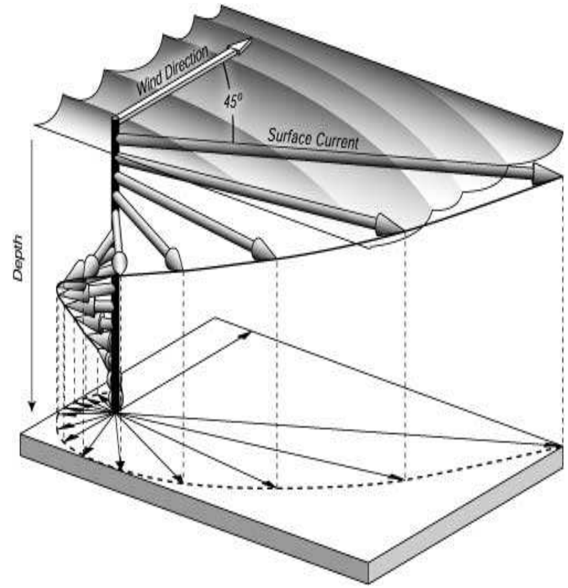
The Navier-Stokes-Coriolis equations admits an explicit stationary solution (u_E, p_E) :

$$
\begin{aligned}
u_E(x_3) &:= (u_E^1(x_3), u_E^2(x_3), 0) \\
p_E(x_2) &:= -\omega u_\infty x_2
\end{aligned}
$$

with

$$
\begin{aligned}
u_E^1(x_3) &:= u_\infty(1 - e^{-\frac{x_3}{\delta}} \cos(\tfrac{x_3}{\delta})) \\
u_E^2(x_3) &:= u_\infty e^{-\frac{x_3}{\delta}} \sin(\tfrac{x_3}{\delta}),
\end{aligned}
$$

where $\delta := (\frac{2\nu}{|\omega|})^{1/2}$ thickness of boundary layer



- stationary solution goes back to swedish oceanograph V. Ekman, 1905
- in his honour : Ekman spiral
- study here : stability properties of Ekman spiral
- many more examples : quasigeostrophic equations, ...

Deterministic Perturbations

Let (u, p) be a solution of Navier-Stokes-Coriolis system in halfspace and set

$$v := u - u_E, \qquad q := p - p_E$$

Then (v, q) satisfies the equation

$$\begin{aligned} v_t - \Delta v + \omega e_3 \times v + (u_E \cdot \nabla)v + v_3 \frac{\partial u_E}{\partial x_3} + v \cdot \nabla v + \nabla q &= 0, \ x \in \mathbb{R}^3_+, t > 0 \\ \mathrm{div}\ v &= 0, \ x \in \mathbb{R}^3_+, t > 0 \\ v(t, x_1, x_2, 0) &= 0, \ x_1, x_2 \in \mathbb{R}, t > 0 \\ v(0, x) &= u_0(x) \end{aligned}$$

We say that u_E is nonlinearly stable if above equation admits a "global solution" v such that $v(t) \to 0$ as $t \to \infty$ in a certain sense.

Stochastic Perturbations

Consider stochastic analogue in layer $D := \mathbb{T}^2 \times (0, b)$

$$\begin{cases} du_t = [\nu \Delta u_t - \omega(e_3 \times u_t) - (u_t \cdot \nabla) u_t + \nabla p_t] dt + dW_t \\ \operatorname{div} u_t = 0 \\ u_t(x_1, x_2, 0) = 0 \\ u_t(x_1, x_2, b) = e_1 \cdot u_b \end{cases}$$

- $(W_t)_{t \geq 0}$ is H-valued Q-Wiener process, where $H := L^{2,per}_{\sigma}(D)$ defined on stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$

Consider stochastic perturbations u_t of u_b^E, i.e.

$$v_t = u_t + u_b^E, \qquad q_t = p_t + p_b^E$$

Mathematical Analysis I : Navier-Stokes-Coriolis

Strategy for strong well-posedness for Navier-Stokes :

- write equations of Navier-Stokes as Evolution Equation

$$u'(t) - Au(t) = -P[u(t) \cdot \nabla)u(t)$$

  in Banach space $L^p_\sigma(\Omega)$, where
  - $A = P\Delta$, Stokes operator
  - P, Helmholtz projection
- rewrite evolution equation as integral equation

$$u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A} P[(u(s) \cdot \nabla)u(s)]ds$$

- solve integral equation via fixed point methods or iteration scheme
- Find function space F in which iteration scheme
  - $u_1(t) = e^{tA}u_0$
  - $u_{n+1}(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A} P[(u_n(s)\nabla)u_n(s)]ds$ converges.
- important : properties of Stokes operator and Stokes semigroup

Unique, Strong solutions for Equations of Navier-Stokes

Assume $\Omega \subset \mathbb{R}^3$ bounded, $\partial\Omega$ smooth

- Fujita-Kato : if either $u_0 \in D(A)^{1/4}$ or interval of existence for T is sufficiently small, then there exists a unique, strong solution on $[0, T)$.
- in particular : L^2-situation : $u_0 \in \dot{H}^{1/2}$
- Extension of iteration schema on scaling invariant function spaces
- key results by Y. Giga '86, T. Kato : $u_0 \in L^p_\sigma(\Omega)$ for $p \geq 3$
- Cannone-Meyer : Well-posedness for $u_0 \in B^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$
- Koch-Tataru : Well-posedness for $u_0 \in BMO^{-1}(\mathbb{R}^3)$
- Bourgain-Pavlovic : Ill-posedness for $u_0 \in B^{-1}_{\infty,\infty}(\mathbb{R}^3)$, i.e. solution operator $u_0 \mapsto u(t)$ is not continuous with respect to $\| \cdot \|_{B^{-1}_{\infty,\infty}}$
- global strong solution provided $n = 2$

Navier-Stokes-Coriolis

Recall

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \Omega e_3 \times u + \nabla p &= f, \quad \text{in } [0,T] \times \mathbb{R}^3 \\ \text{div } u &= 0, \quad \text{in } [0,T] \times \Omega \\ u(0) &= u_0, \quad \text{in } \Omega \end{aligned}$$

- Babenko, Mahalov, Nikolenco : pioneering result on global well-posedness for large data provided Ω is large enough
- global well-posedness result Chemin, Desjardins, Gallagher, Grenier :
- let $u_0 \in H^{1/2}(\mathbb{R}^3)$ with div $u_0 = 0$. Then exists $\Omega_0 > 0$ such that for all $\Omega \geq \Omega_0$ the (NSC)-equation admits a unique, global mild solution
- surprising : no smallness condition for u_0
- proof relies on dispersive estimates for linear semigroup $e^{tA_{SCE}}$

$$e^{tA_{SCE}} f = e^{i\Omega t \frac{R_3}{\Delta^{1/2}}} [e^{t\Delta (I+R)f}] + e^{-i\Omega t \frac{R_3}{\Delta^{1/2}}} [e^{t\Delta (I-R)f}]$$

Strichartz Estimates by Koh, Lee, Takada

Let $2 \leq q \leq \infty$, $2 \leq r < \infty$ satisfy $1/q + 1/r \leq 1/2$. Then

$$\|e^{i\Omega t \frac{R_3}{\Delta^{1/2}}} f\|_{L^q(0,\infty);L^r(\mathbb{R}^3))} \leq C|\Omega|^{-1/q}\|f\|_{H^{3/2-3/r}}$$

- Remark : no smoothing in spatial variable
- Further results : global solutions uniform in Ω : $u_0 \in FM_0^{-1}(\mathbb{R}^3)$, $u_0 \in H^{1/2}(\mathbb{R}^3)$, ...

Open Questions

- dispersive estimates for domains with boundaries

Mathematical Analysis II : Primitive Equations

- '92-'95 : full primitive equations introduced by Lions, Temam and Wang, existence of a global weak solution for $a \in L^2$.
- Uniqueness question seems to be open
- '01 : Guillén-González, Masmoudi, Rodiguez-Bellido : existence of a unique, local, strong solution for $a \in H^1$
- '07, Cao and Titi : breakthrough result : existence of a unique, global strong solution for arbitrary initial data $a \in H^1$
- Aim : show existence of a unique, global strong solution to primitive equations for data a having less differentiability properties than H^1.

Strategy of L^p-Approach

- solution of the linearized equation is governed by an analytic semigroup T_p on the space X_p
- X_p is defined as the range of the hydrostatic Helmholtz projection $P_p : L^p(\Omega)^2 \to L^p_{\overline{\sigma}}(\Omega)^2$
- This space corresponds to solenoidal space $L^p_\sigma(\Omega)$ for Navier-Stokes equations
- generator of T_p is $-A_p$ called the hydrostatic Stokes operator.
- rewrite primitive equations as

$$\begin{cases} v'(t) + A_p v(t) = P_p f(t) - P_p(v \cdot \nabla_H v + w\partial_z v), & t > 0, \\ v(0) = a. \end{cases}$$

- consider integral equation

$$v(t) = e^{-tA_p}a + \int_0^t e^{-(t-s)A_p}\big(P_p f(s) + F_p v(s)\big)\, ds, \qquad t \geq 0,$$

where $F_p v := -P_p(v \cdot \nabla_H v + w\partial_z v)$

Strategy of L^p-approach

- show that v is unique, local, strong solution, i.e. $v \in C^1((0,T^*];X_p) \cap C((0,T^*];D(A_p)),\ p \in (1,\infty)$
- Hence, one ontains existence of a unique, global, strong solution for arbirtrary $a \in [X_p, D(A_p)]_{1/p}$ for $1 < p < \infty$ provided
- $\sup_{0 \le t \le T} \|v(t)\|_{H^2(\Omega)}$ is bounded by some constant $B = B(\|a\|_{H^2(\Omega)}, T)$ for any $T > 0$.
- proof of global H^2-bound for v
- in addition : $\|v(t)\|_{H^2(\Omega)}$ is decaying exponentially as $t \to \infty$.
- Recent Theorem :
  Let $p \in (1,\infty)$, $a \in V_{1/p,p}$ and $f \equiv 0$. Then there exists a unique, strong global solution (v,π) to primitive equations within the regularity class

$$v \in C^1((0,\infty);L^p(\Omega)^2) \cap C((0,\infty);W^{2,p}(\Omega)^2),\ \pi \in C((0,\infty);W^{1,p}(G) \cap L_0^p(G)).$$

Moreover, the solution (v,π) decays exponentially, i.e. there exist constants $M, c, \tilde{c} > 0$ such that

$$\|\partial_t v(t)\|_{L^p(\Omega)} + \|v(t)\|_{W^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(G)} \le M t^{-\tilde{c}} e^{-ct}, \quad t > 0.$$

Open questions

- rough data ?, $a \in L^\infty$?
- realistic domains : domains with „islands“
- fluid-structure interaction : iceberg swimming in hydrostatic fluid
- moving iceberg is melting, Stefan type problems

Mathematical Analysis III : Ekman Layers

- Consider above perturbated equation as evolution equation in $L^p_\sigma(\mathbb{R}^3_+)$
- Rewrite perturbated equation in $L^p(\mathbb{R}^3_+)$ as

$$\begin{aligned} u_t + A_{SCE}u + P(u \cdot \nabla u) &= 0, \qquad t > 0, \\ u(0) &= u_0 \end{aligned}$$

$$\begin{aligned} A_S u &:= P\Delta u, && \text{Stokes Operator} \\ A_C u &:= P\omega e_3 \times u, && \text{Coriolis Operator} \\ A_E u &:= P[u_E \cdot \nabla)u + u_3 \frac{\partial u_E}{\partial x_3}] && \text{Ekman Operator} \\ A_{SCE} &:= A_S + A_C + A_E, && \text{Stokes-Coriolis-Ekman} \end{aligned}$$

- Define Reynolds number as $R := \frac{u_\infty \delta}{\nu}$
- Stability problem : there exists critical Reynolds number R_c such that
  - $R < R_c \Longrightarrow$ solution is stable
  - $R > R_c \Longrightarrow$ solution is unstable
  - construct suitable weak solution which allows to deduce asymptotic properties

Idea of construction of such a weak solutions

Consider Yosida approximation by operator J_k

$$J_k := k(k - A_{SCE})^{-1}, \quad k \in \mathbb{N}.$$

and set

$$w_{0k} := J_k w_0 \text{ and } F_k w := -P(J_k w \cdot \nabla) w$$

and construct approximate solutions w_k for small t by solving the integral equation

$$w_k(t) = e^{tA_{SCE}} w_{0k} + \int_0^t e^{(t-s)A_{SCE}} F_k w_k(s) ds.$$

in the Banach space $X := C([0, T]; D(A_{SCE}^{1/2}))$

Construction in four steps :

- Step 1 : existence of an approximate solution for small t
- Step 2 : existence of an approximate solution for given large $T > 0$
- Step 3 : extract weakly converging subsequence
- Step 4 : subsequence fulfills perturbed equations

Stability Results for Ekman spiral

Assume Reynolds number $R = \frac{u_\infty \delta}{\nu}$ is small. Then

- for all $w_0 \in L^2_\sigma(\mathbb{R}^3_+)$ there exists a weak solution to perturbed equation with $w(0) = w_0$ satisfying

$$\lim_{T \to \infty} \int_T^{T+1} ||w(s)||_{H^1} ds = 0$$

- Giga et al : stability criteria for non-decaying perturbations in other function spaces
- determine how $\sigma(A_{SCE})$ changes with Reynolds number
- instabilty results in above norm ?
- Ekman spirals over spheres ?

Mathematical Analysis IV : Stratified Fluids

Recall

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - \nu\Delta v + \nabla\pi - \Omega e_3 \times v &= \theta e_3 + g, \qquad \text{in } \mathbb{R}^3 \times (0, T), \\ \partial_t \theta + (v \cdot \nabla)\theta - \kappa\Delta v &= N^2 v_3 + h, \qquad \text{in } \mathbb{R}^3 \times (0, T), \\ \operatorname{div} v &= 0, \qquad \text{in } \mathbb{R}^3 \times (0, T). \end{cases}$$

- assume g, h are peridoc with period T
- do geophysical equations allow for periodic solutions if forces are periodic ?
- Navier-Stokes : yes, for f small
- primitive : yes, for f large
- rotating stratified fluids, yes for large f if rotation is large
- use again disperive effect of rotation
- periodic solutions for primitive equation if Δ is replaced by Δ_H ?

Primitive : Periodic Solutions for Large Forces

Aims :

- show existence of strong time-periodic solutions for arbitrary (time-periodic) $f \in L^2(0, \mathcal{T}, L^2(\Omega))$, without assuming any smallness condition on f
- Consequence : analogous result for steady-state solutions

Approach based on three steps :

- construct a suitable weak time-periodic solution v by combining classical Galerkin's method with Brouwer's fixed point theorem.
- show existence of a unique, strong solution u to the initial-value problem for arbitrary $f \in L^2(0; \mathcal{T}; L^2(\Omega))$ and a in a subspace of $H^1(\Omega)$
- look at v as a weak solution to the initial-value problem, employ weak-strong uniqueness argument : This yields $v \equiv u$

Weak and Strong Periodic Solutions

v is a weak T-periodic solution provided

- $v \in C(J; L^2(\Omega)) \cap L^2(J; H^1(\Omega))$ is a weak solution
- v satisfies strong energy inequality

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2 + 2\int_s^t (f(\tau), v(\tau))d\tau$$

- $v(t+T) = v(T)$ for all $t \geq 0$

A weak T-periodic solution v is strong if in addition $v \in C(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega))$

Proposition : Let $f \in L^2(J; L^2(\Omega))$ be T-periodic. Then there exists at least one weak T-periodic solution v

Proof : Galerkin procedure and Brouwer's fixed point theorem

Periodic Solutions via Weak-Strong Uniqueness

- Let $f \in L^2(J; L^2(\Omega))$ be T-periodic. Then there exists unique global strong solution u for arbitrary large $a \in H^1(\Omega)$
- weak-strong uniqueness theorem : $u = v$
  - Idea of Proof :
  - weak theory : there is $t_0 > 0$ with $v(t_0) \in H^1$
  - take $v(t_0)$ as initial data for strong solution u
  - take u as test function
  - for $w = v - u$ one has

$$\|w(t)\|_2^2 + \int_{t_0}^{t} \|\nabla w(s)\|_2^2 ds \leq C \int_{t_0}^{t} [\|\nabla_H u(s)\|_2^4 + \|\nabla_H u(s)\|_2^2 \|D^2 u(s)\|_2^2] \|w(s)\|_2^2 ds$$

  - blue term in $L^1(t_0, t)$ due to regularity of strong solutions u
  - Gronwall : $w = 0$
- Theorem : primitive equations admit a strong, periodic solution for non small periodic $f \in L^2(J, L^2)$
- Corollary : primitive equations admit a stationary solution for non small periodic $f \in L^2(J, L^2)$