International/Interdisciplinary Seminar on Nonlinear Science
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A variational formulation for dissipative systems

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TOPICS

1 In terms of an optimized control theory, the realized dynamics of physical systems take stationary values of cost functionals.

2 The equations of motion for dissipative systems are obtained by solving stationary problems subject to non-holonomic constraints for entropy.

3 The non-holonomic constraints are determined to satisfy the law of entropy increase, symmetries, and well-posedness.

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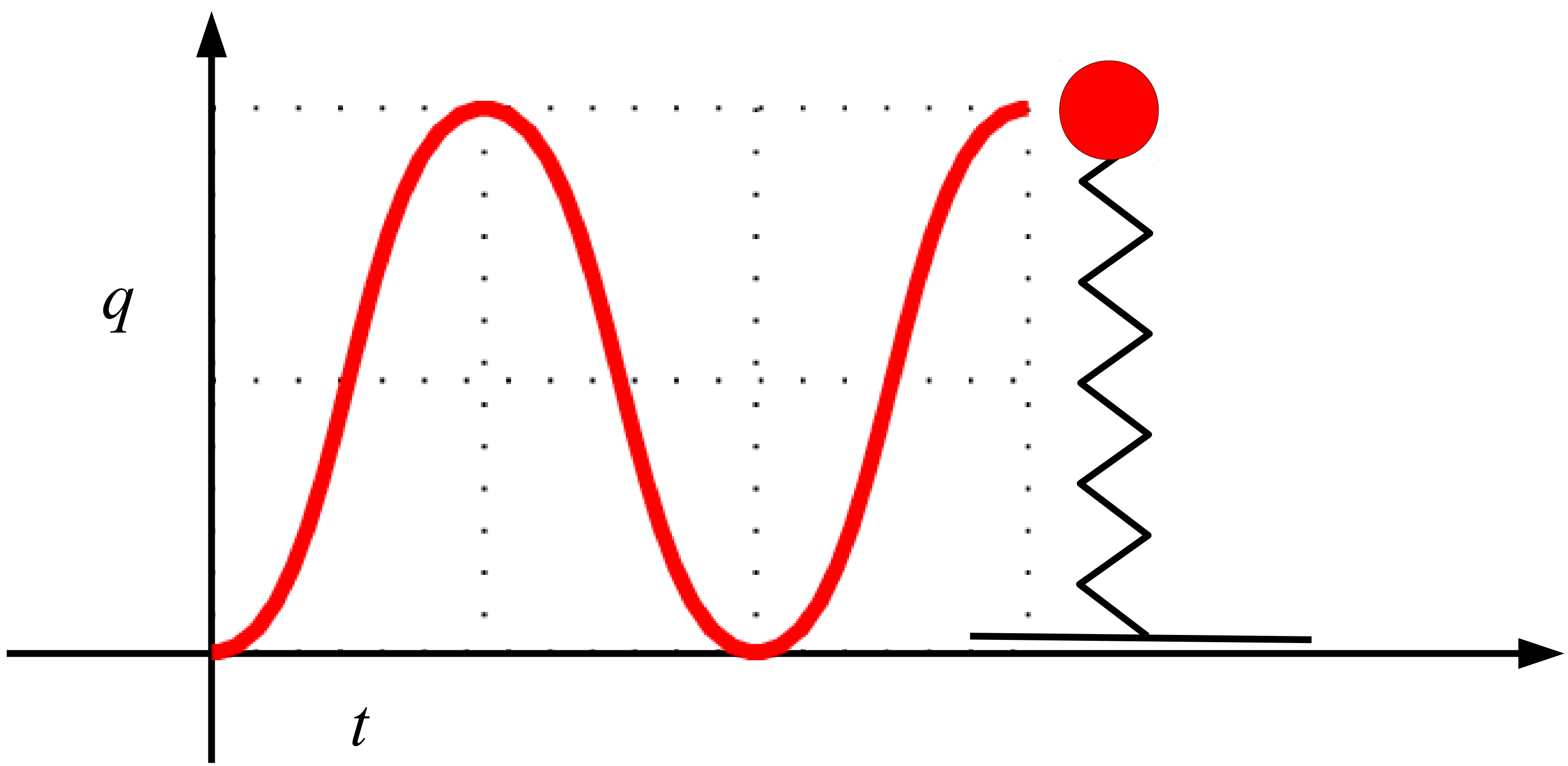
How to describe physics

The time development of a physical system can be described as a curve C in configuration space q and time t.



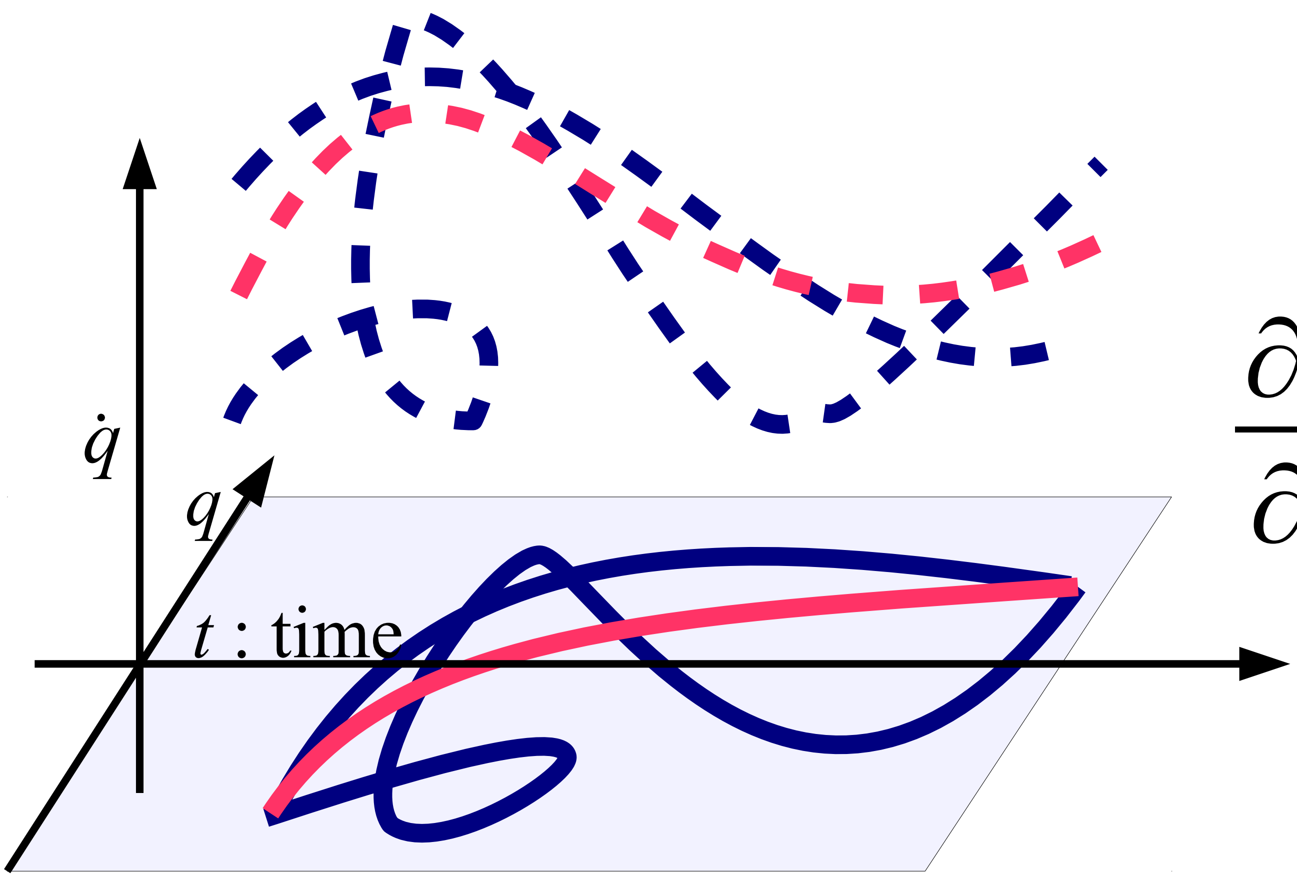
Variational principles

A realized motion gives a stationary value of a functional.

$$\int_C L(q, \dot{q}, t)\, dt$$

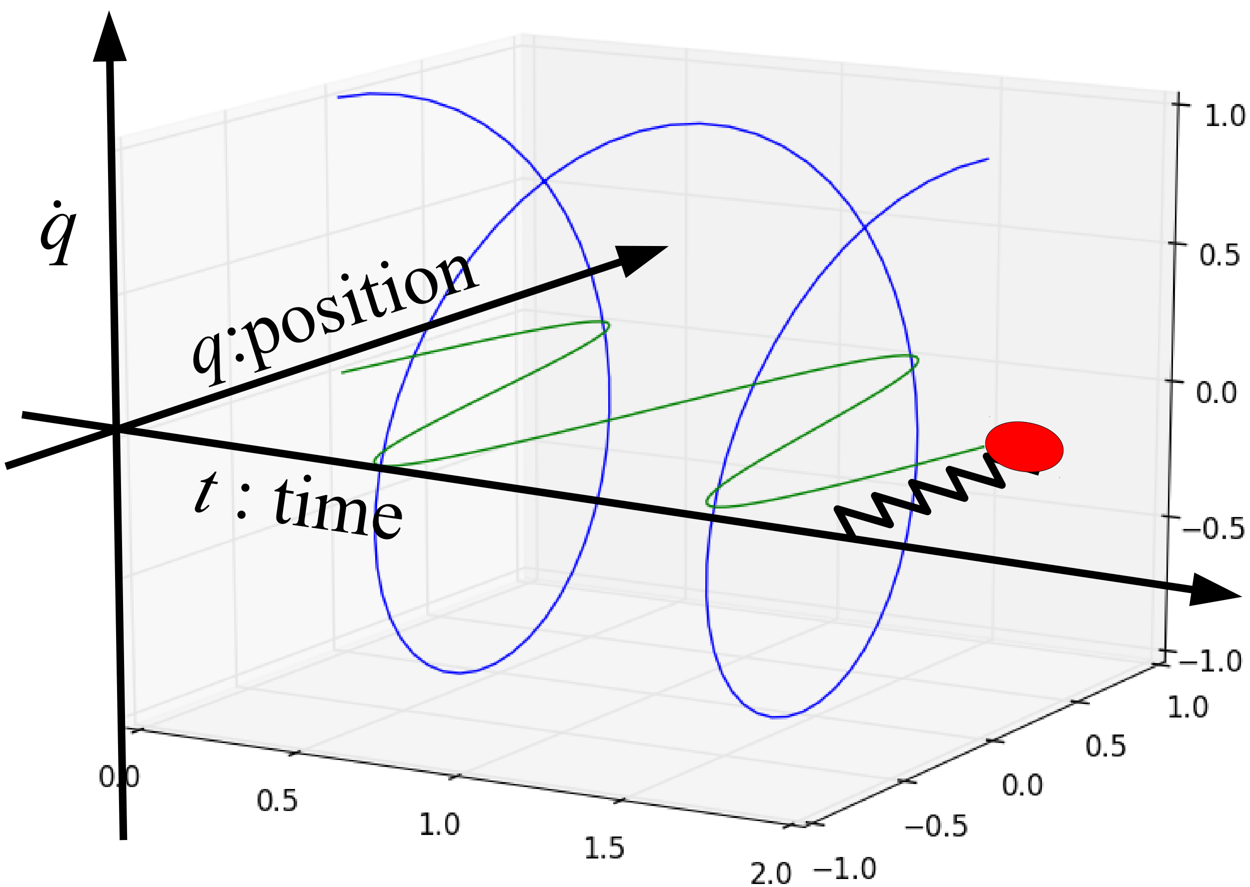
Euler-Lagrange Eq.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$



The dynamics of a harmonic oscillator

Action functional $\int_C \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 \right) dt$



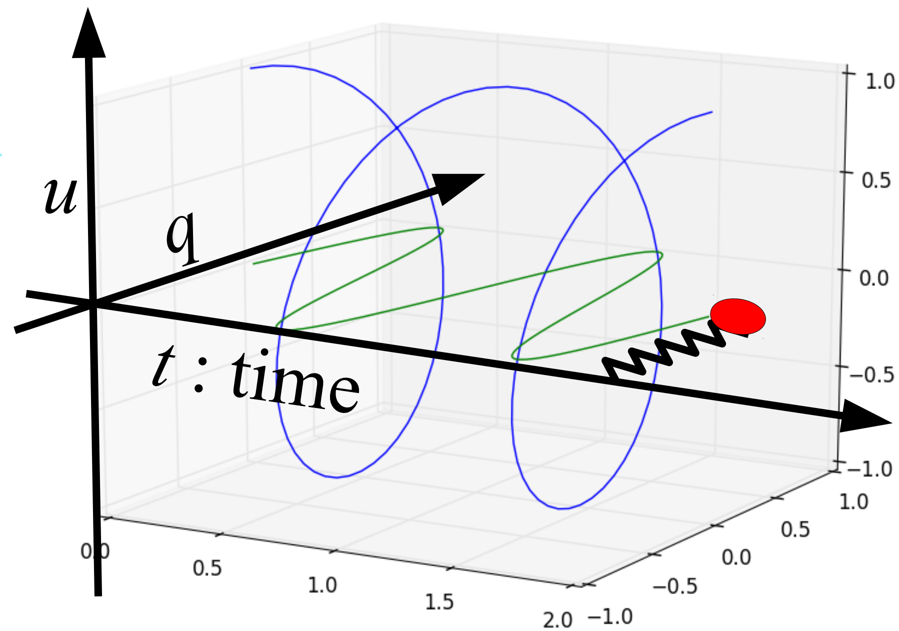
Euler-Lagrange Eq.

$$\ddot{q} + q = 0$$

$$\begin{pmatrix} \dot{q} \\ q \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Variational calculus of an action functional

$$\int_0^1 L(q(\tau), u(\tau)) \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau$$



$$\begin{aligned} 0 &= \lim_{\alpha\to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} L dt \\ &= \int_{C_0} \left[\left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial u} \delta u \right) \frac{dt}{d\tau} + L \frac{d\delta t}{d\tau} \right] d\tau \end{aligned}$$

The relation: $\dfrac{dq}{dt} - u = 0$ yields

$$
\begin{aligned}
0 &= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} p(dq - u dt) \\
&= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} p \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} - u \frac{\mathrm{d}t}{\mathrm{d}\tau} \right) d\tau \\
&= \int_{C_0} p \left(\frac{\mathrm{d}\delta q}{\mathrm{d}\tau} - \delta u \frac{\mathrm{d}t}{\mathrm{d}\tau} - u \frac{\mathrm{d}\delta t}{\mathrm{d}\tau} \right) d\tau \\
&= \int_{C_0} \left(-\frac{dp}{dt} \delta q - p \delta u + \frac{d}{dt}(pu) \delta t \right) \frac{\mathrm{d}t}{\mathrm{d}\tau} d\tau \\
&\quad + \left[p\delta q - pu\delta t \right]_{\tau=0}^{\tau=1} .
\end{aligned}
$$

$$0 = \int_{C_0} \left[\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial u}\delta u \right) \frac{dt}{d\tau} + L\frac{d\delta t}{d\tau} \right] d\tau$$

$$+)\quad 0 = \int_{C_0} \left(-\frac{dp}{dt}\delta q - p\delta u + \frac{d}{dt}(pu)\delta t \right) \frac{dt}{d\tau} d\tau$$

where $p = \partial L / \partial u$

$$0 = \int_0^1 \left[\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial u} \right) \delta q - \frac{d}{dt}\left(L - \frac{\partial L}{\partial u} u \right) \delta t \right] \frac{dt}{d\tau} d\tau$$

Euler-Lagrange eq. Energy conservation

Method of Lagrange multiplier

$$\int_C Ldt + p(dq - udt) = \int_C pdq - \tilde{H}dt$$

$$\tilde{H}(q,p,u) \equiv pu - L(q,u)$$

$$\int_{t(0)}^{t(1)} \left[\left(\frac{dq}{dt} - \frac{\partial \tilde{H}}{\partial p} \right) \delta p - \left(\frac{dp}{dt} + \frac{\partial \tilde{H}}{\partial q} \right) \delta q - \frac{\partial \tilde{H}}{\partial u} \delta u + \frac{d\tilde{H}}{dt} \delta t \right] dt = 0$$

$$H(p,q) \equiv \tilde{H}(p,q,u^*(p,q))$$

where $u^*(p,q)$ is the solution of $\dfrac{\partial \tilde{H}}{\partial u} = 0$.

Hamilton's eq. $$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

A harmonic oscillator

$$L(q,u)=\frac{1}{2}u^2-\frac{1}{2}q^2, \qquad \widetilde{H}(p,q,u)=pu-L(q,u)$$

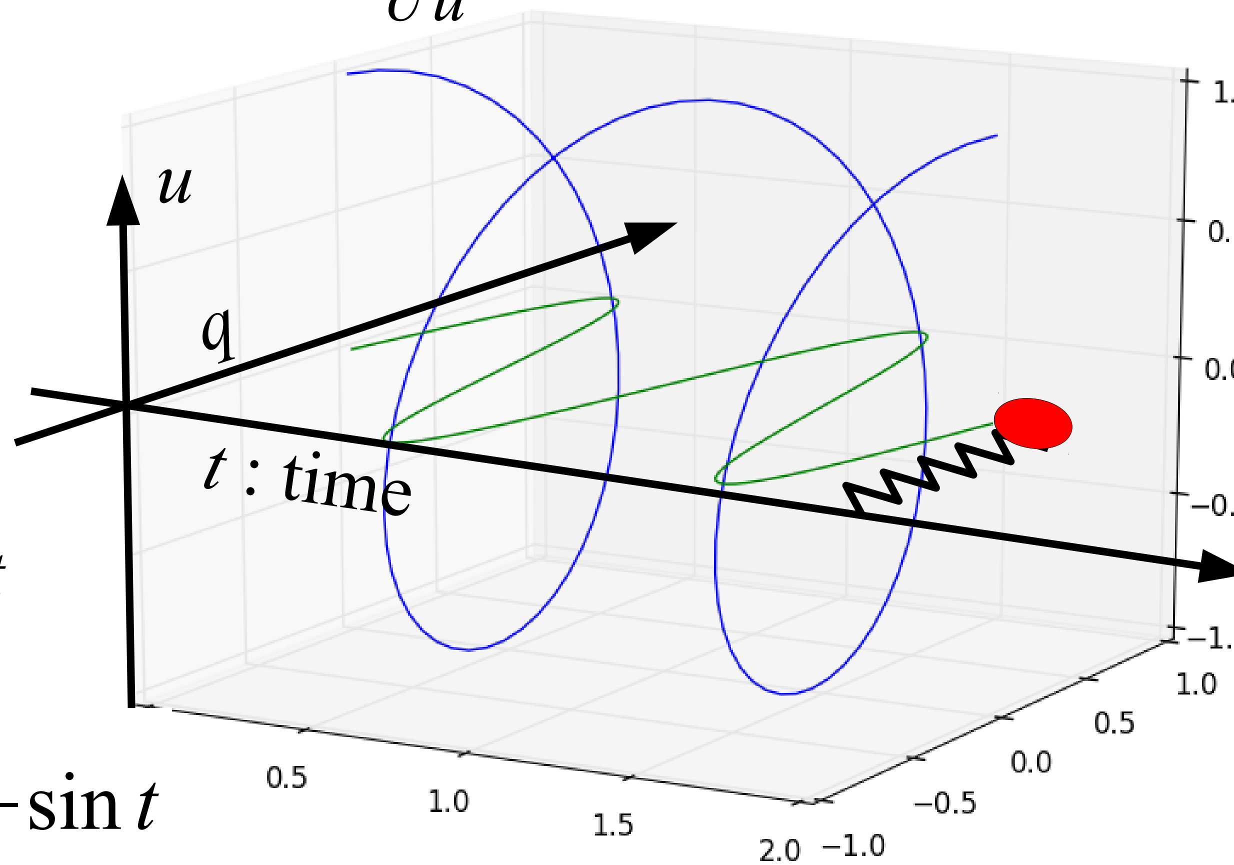
Put $u^*(p,q)=\frac{\partial \widetilde{H}}{\partial u}=p$

$$H(p,q)=\frac{1}{2}p^2+\frac{1}{2}q^2$$

Hamilton's equations

$$\frac{dq}{dt}=\frac{\partial H}{\partial p}=p=\cos t$$

$$\frac{dp}{dt}=-\frac{\partial H}{\partial q}=-q=-\sin t$$



Pontryagin’s maximum principle (PMP)

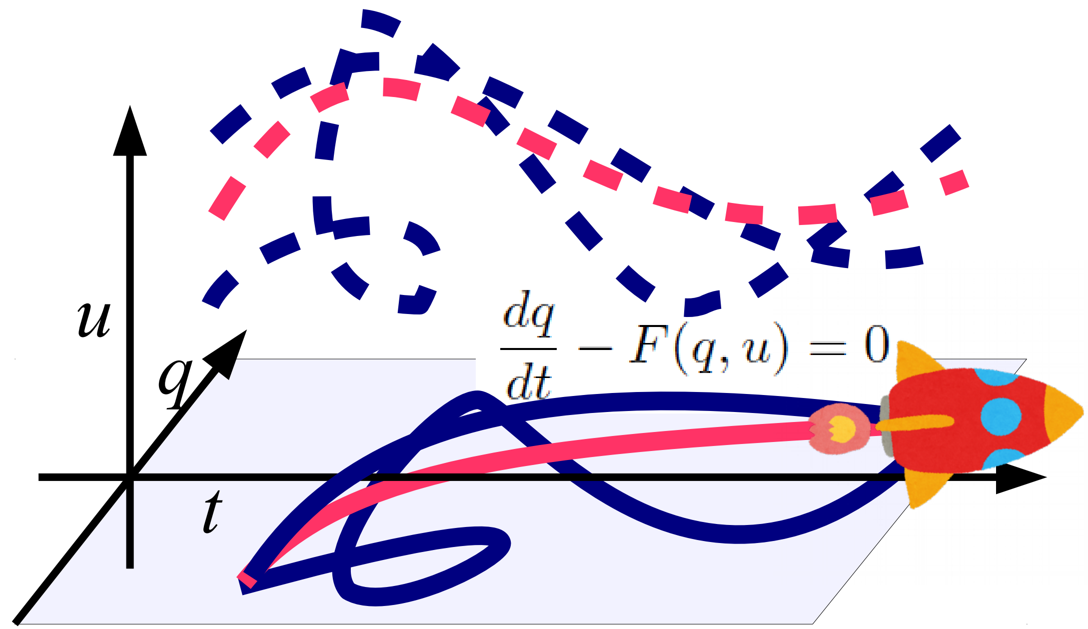
Find an optimized control u^* which gives the stationary value of a cost functional, $\int_C L dt$.

$$\int_C p dq - \tilde{H} dt$$

where $\tilde{H}(p,q,u) = pF(q,u) - L(q,u)$

Hamilton’s equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$



where

$$H(p,q) \equiv \tilde{H}(p,q,u^*(p,q))$$

$u^*(p,q)$ is the solution of $\frac{\partial \tilde{H}}{\partial u} = 0$.

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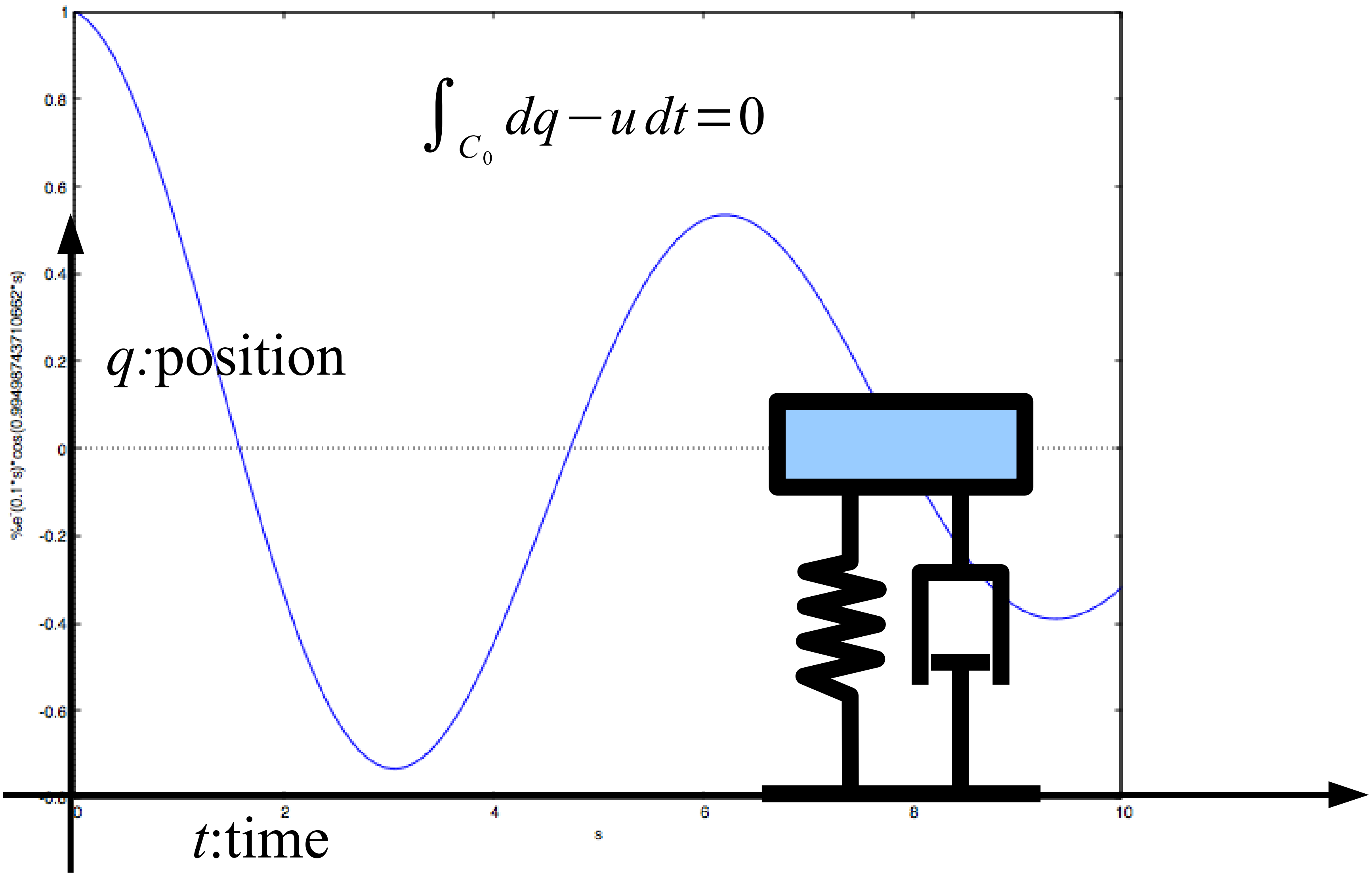
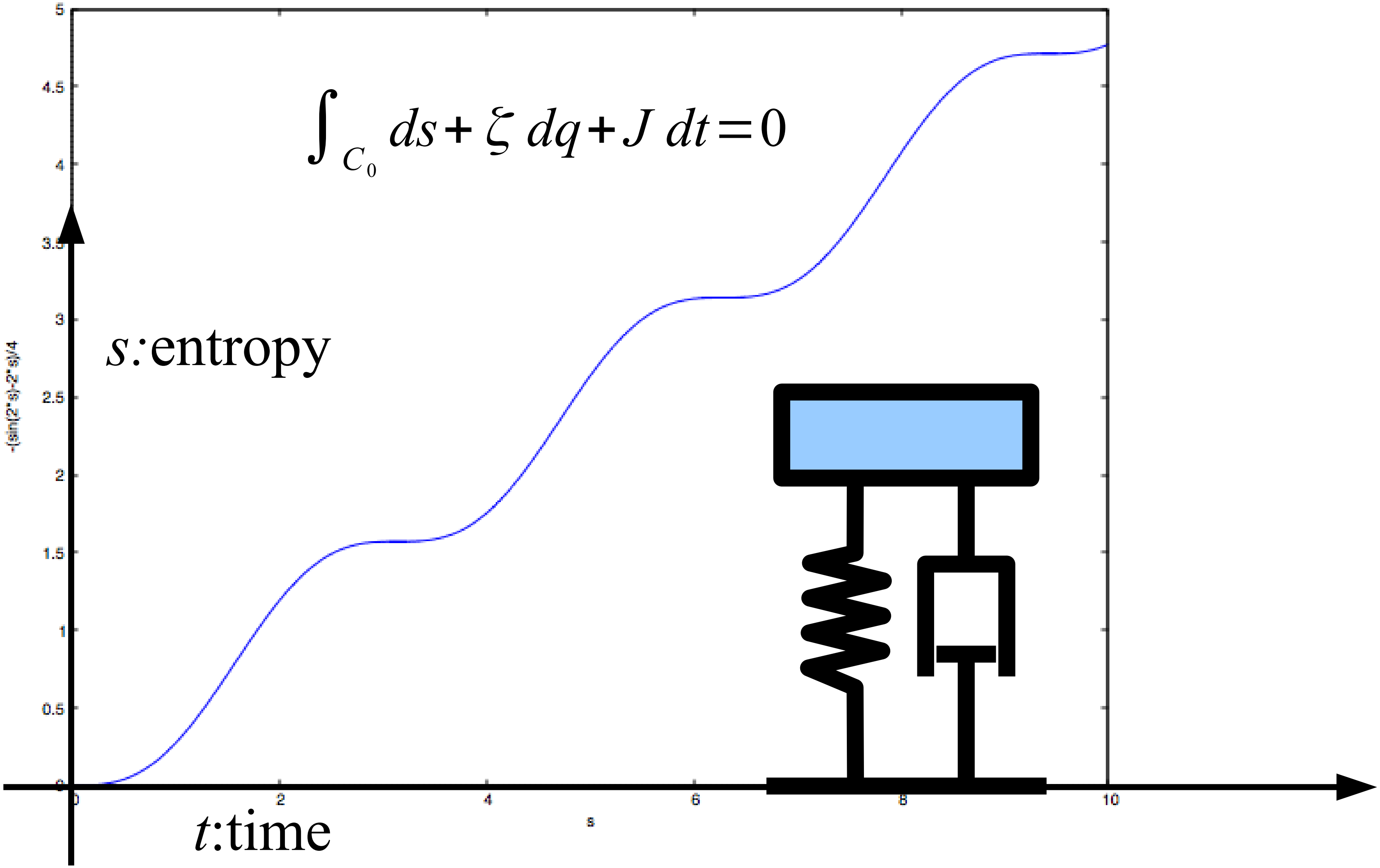
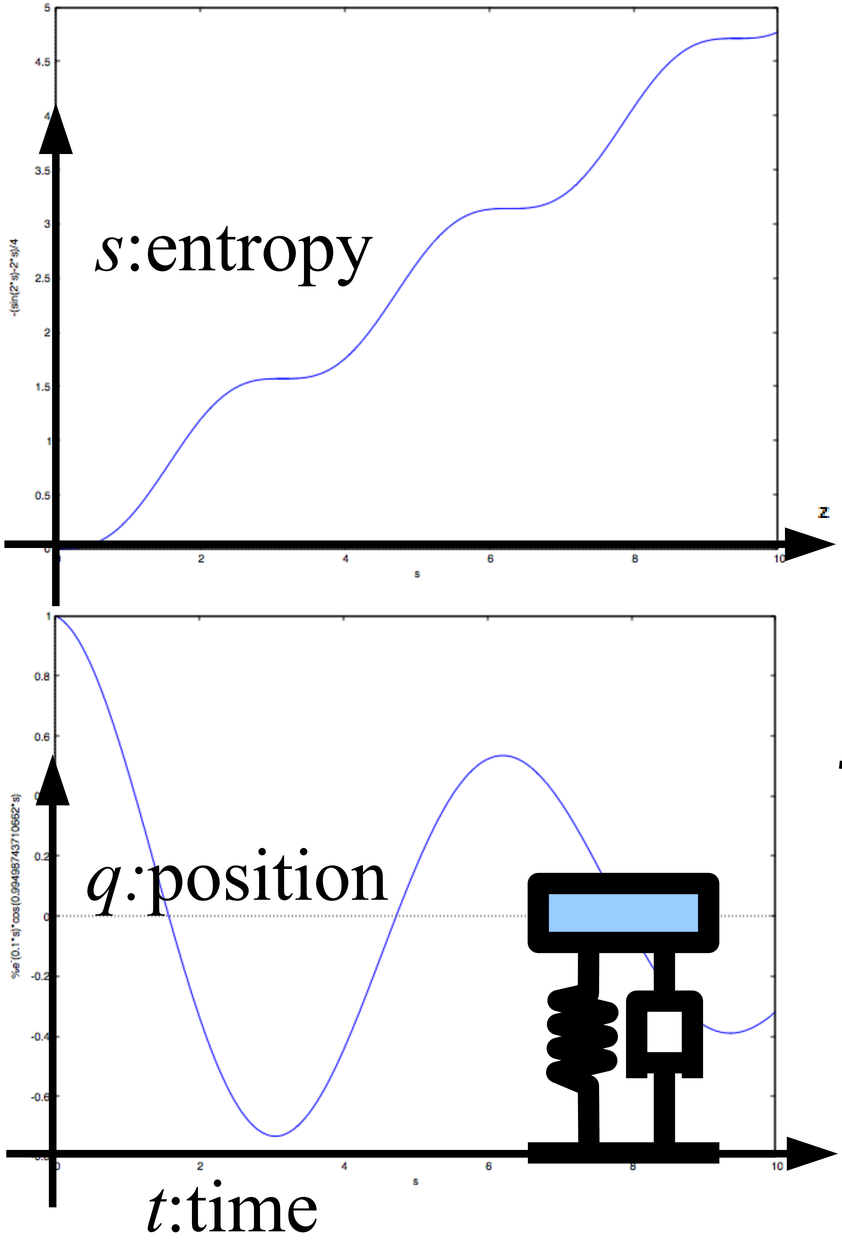
A dissipative system



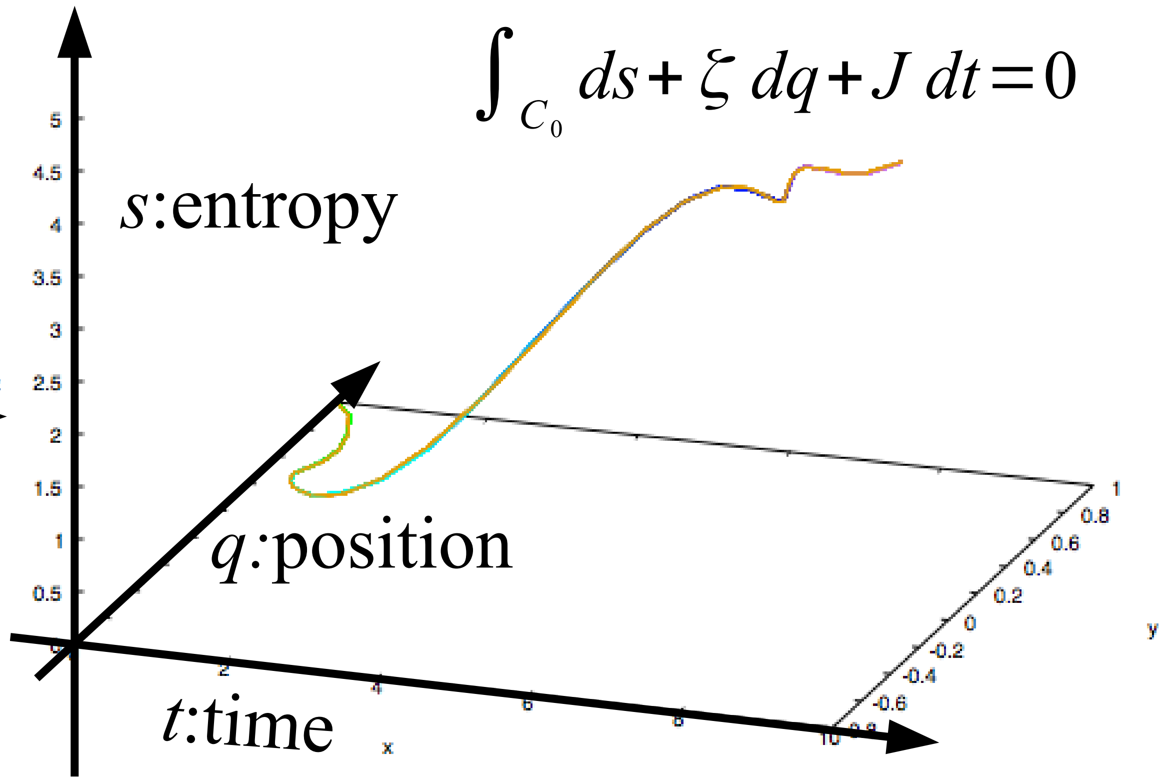
A dissipative system



A dissipative system



$$\int_{C_0} ds + \zeta\, dq + J\, dt = 0$$

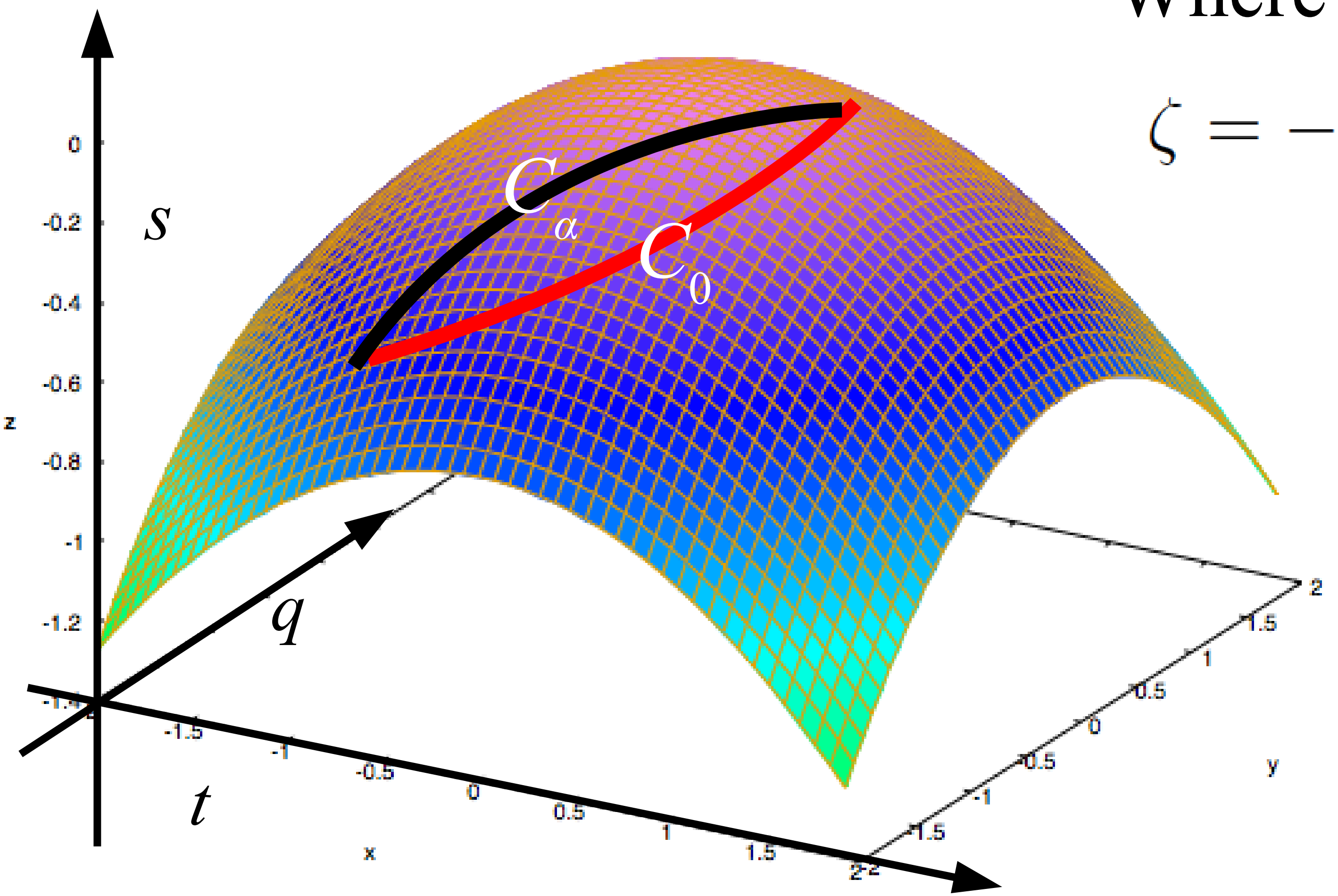


A holonomic constraint

$$\int_{C_0} ds + \zeta\, dq + J\, dt = 0 \quad \Rightarrow \quad \int_{C_0} [s - g(q,t)]\, dt = 0$$

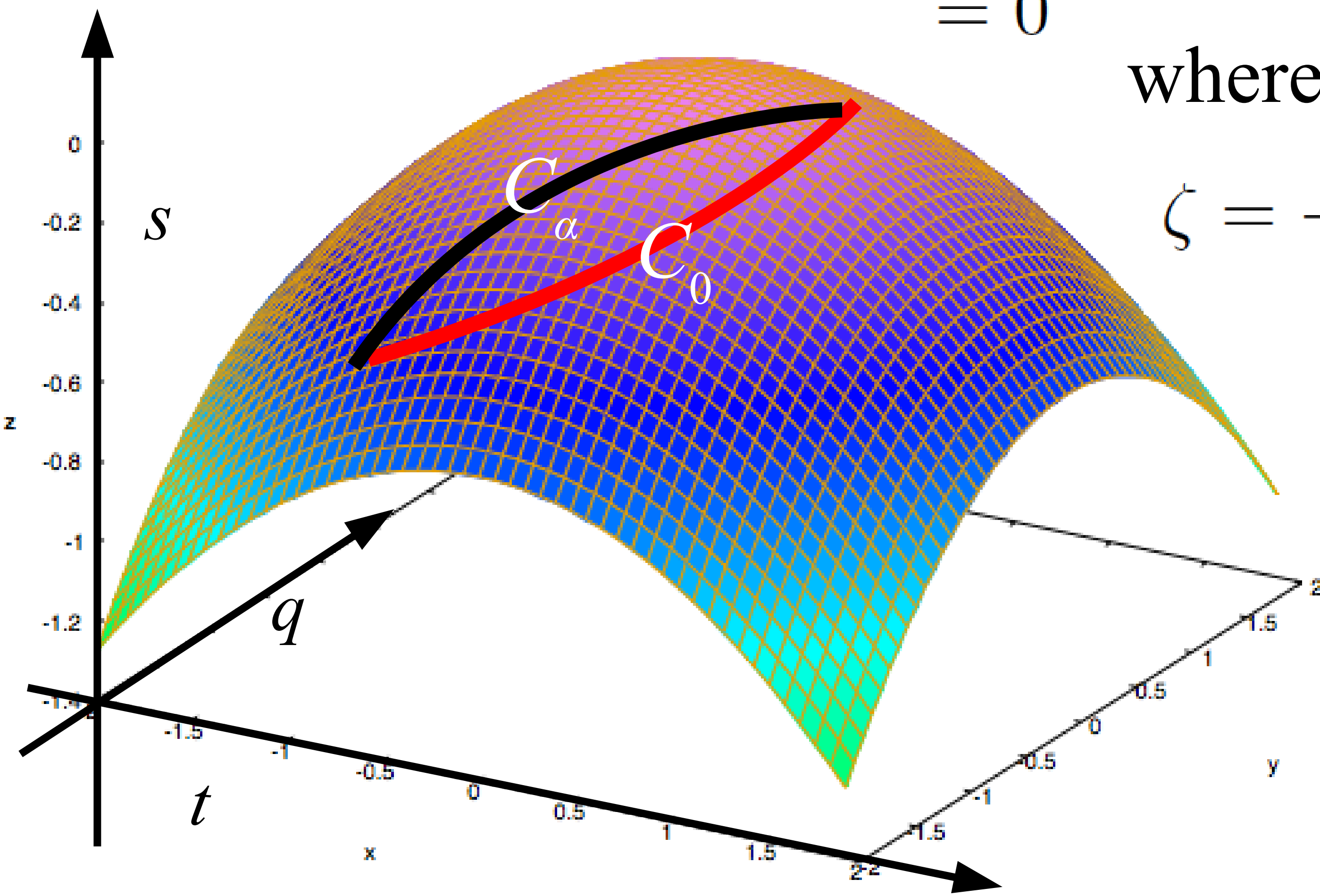
Where

$$\zeta = -\frac{\partial g}{\partial q}, \;\; J = -\frac{\partial g}{\partial t}.$$



A holonomic constraint

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_{C_\alpha - C_0} T\left[s - g(q,t)\right]dt = \int_{C_0} T\left(\delta s + \zeta\delta q + J\delta t\right)dt$$
$$= 0$$



where

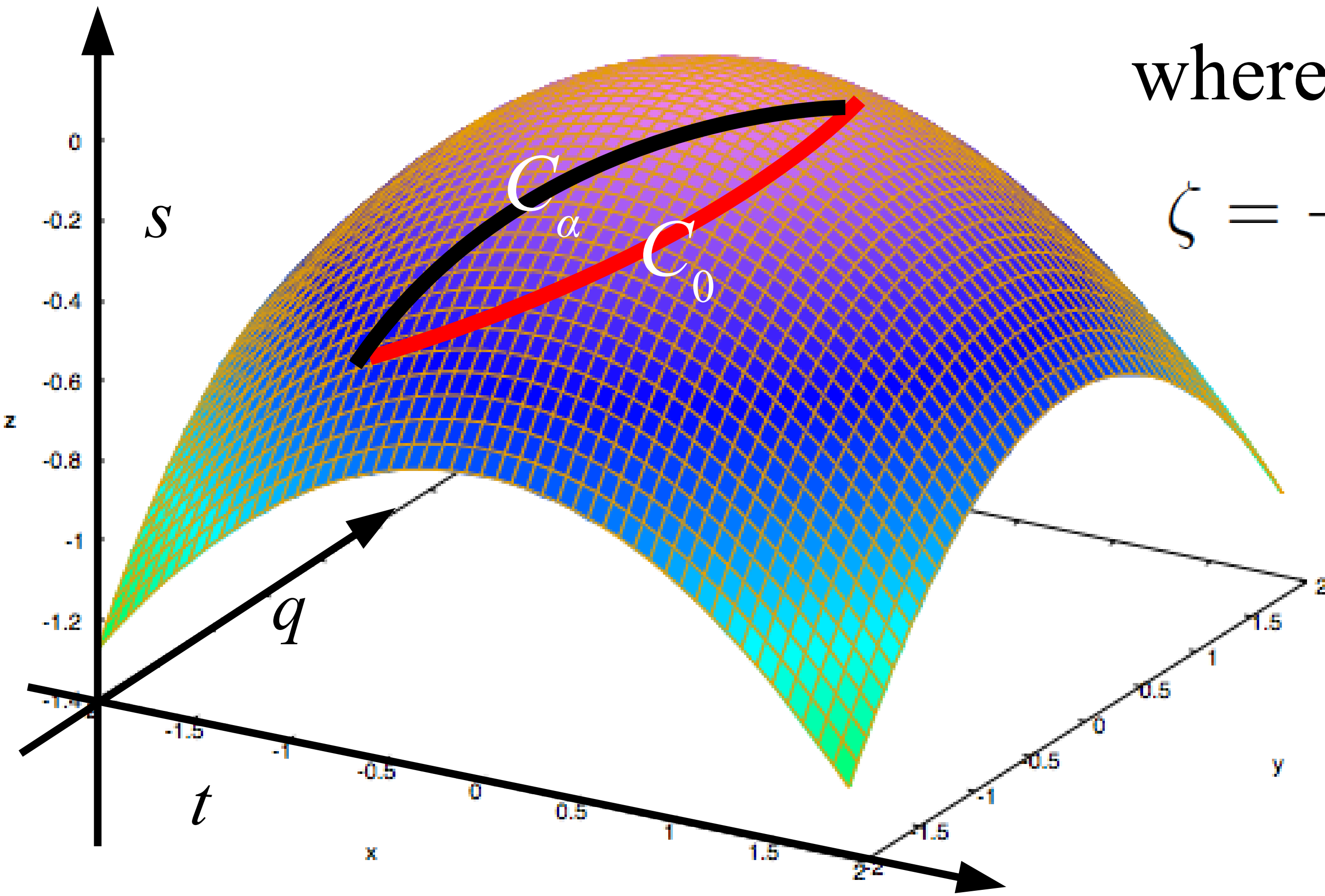
$$\zeta = -\frac{\partial g}{\partial q},\;\; J = -\frac{\partial g}{\partial t}.$$

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} p dq - \tilde{H} dt = 0$$

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} T \left[s - g(q,t) \right] dt = 0$$

where

$$\zeta = -\frac{\partial g}{\partial q}, \; J = -\frac{\partial g}{\partial t}.$$



$$\int_{C_0}\left[\left(\frac{dq}{dt}-\frac{\partial \tilde{H}}{\partial p}\right)\delta p-\left(\frac{dp}{dt}+\frac{\partial \tilde{H}}{\partial q}\right)\delta q-\frac{\partial \tilde{H}}{\partial s}\delta s-\frac{\partial \tilde{H}}{\partial u}\delta u+\frac{d\tilde{H}}{dt}\delta t\right]dt=0$$

$$+)\qquad \int_{C_0} T\left(\delta s+\zeta\delta q+J\delta t\right)dt=0$$

where $T=\dfrac{\partial \tilde{H}}{\partial s}$, $f\equiv T\zeta$, and $Q\equiv TJ$.

$$\int_{C_0}\left[\left(\frac{dq}{dt}-\frac{\partial \tilde{H}}{\partial p}\right)\delta p-\left(\frac{dp}{dt}+\frac{\partial \tilde{H}}{\partial q}-f\right)\delta q-\frac{\partial \tilde{H}}{\partial u}\delta u\right.$$
$$\left.-\left(\frac{\partial \tilde{H}}{\partial s}-T\right)\delta s+\left(\frac{d\tilde{H}}{dt}+Q\right)\delta t\right]dt=0,$$

Method of Lagrange multiplier

$$\int_C p\,dq - \left[\tilde{H} - T(s-g)\right] dt$$

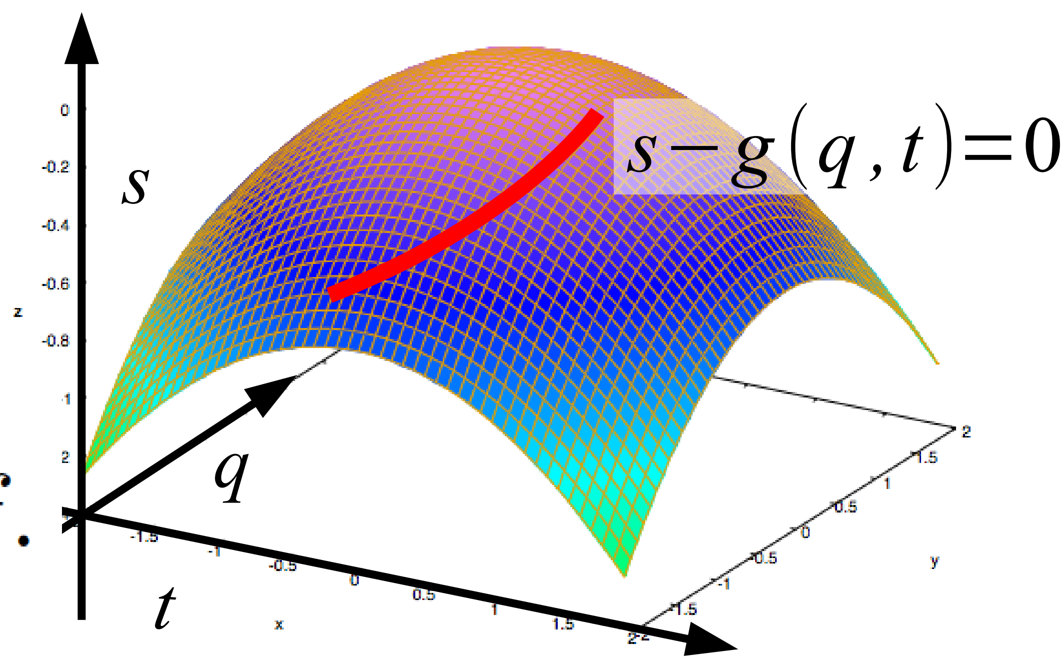
$$H(p,q,s) \equiv \widetilde{H}(p,q,s,u^*(p,q,s))$$

where $u^*(p,q,s)$ is the solution of $\dfrac{\partial \widetilde{H}}{\partial u} = 0.$

Equations of motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p},$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} + f.$$

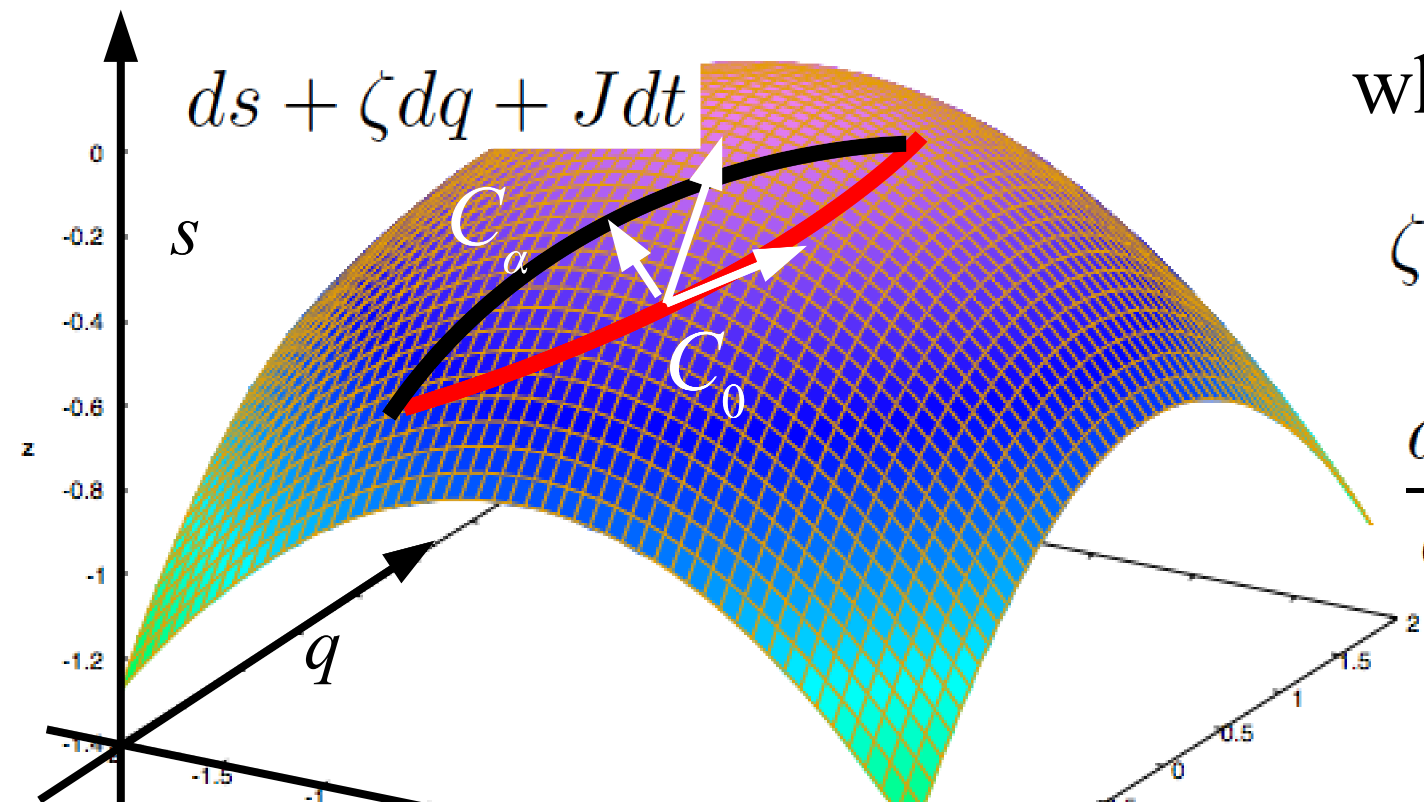


A holonomic constraint

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} T\left[s - g(q,t)\right] dt = \int_{C_0} T\left(\delta s + \zeta \delta q + J \delta t\right) dt$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_{C_\alpha - C_0} \Lambda(ds + \zeta dq + J dt)$$

$$= 0$$

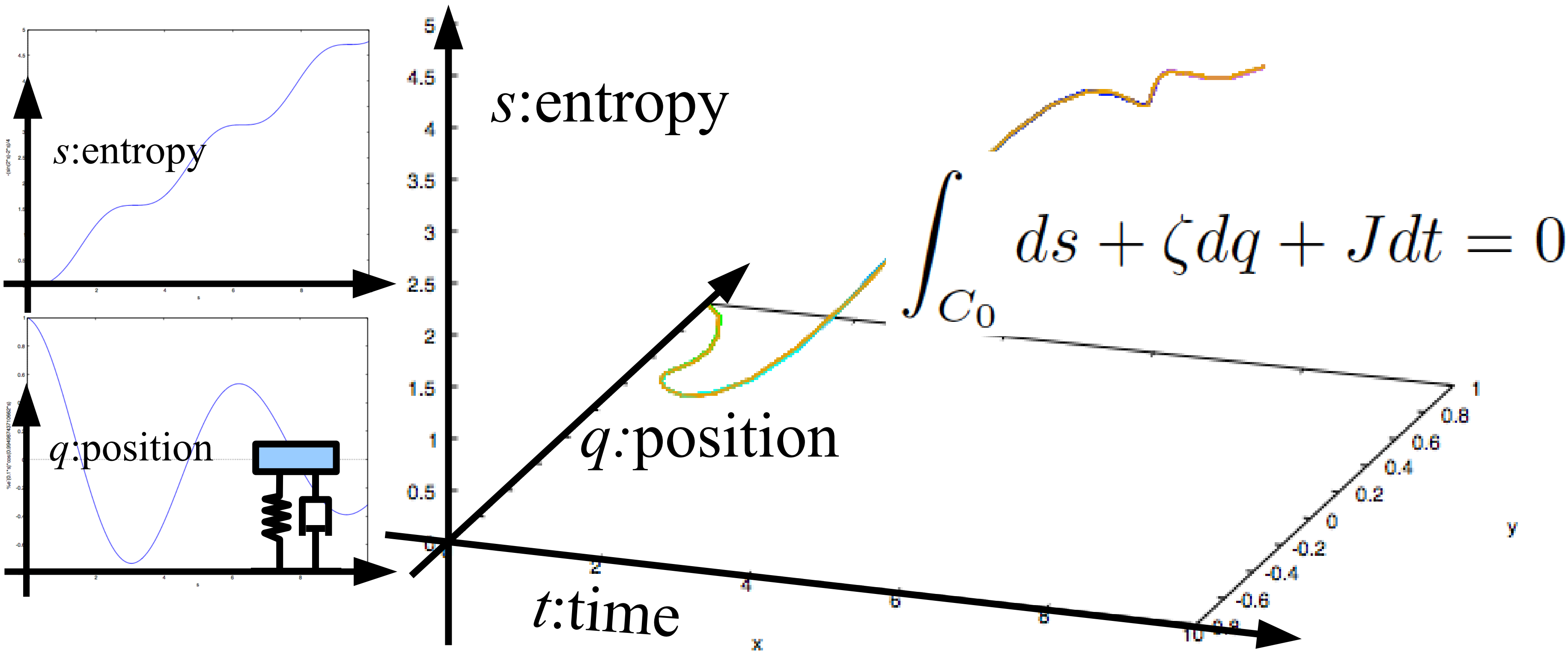


where

$$\zeta = -\frac{\partial g}{\partial q}, \quad J = -\frac{\partial g}{\partial t}.$$

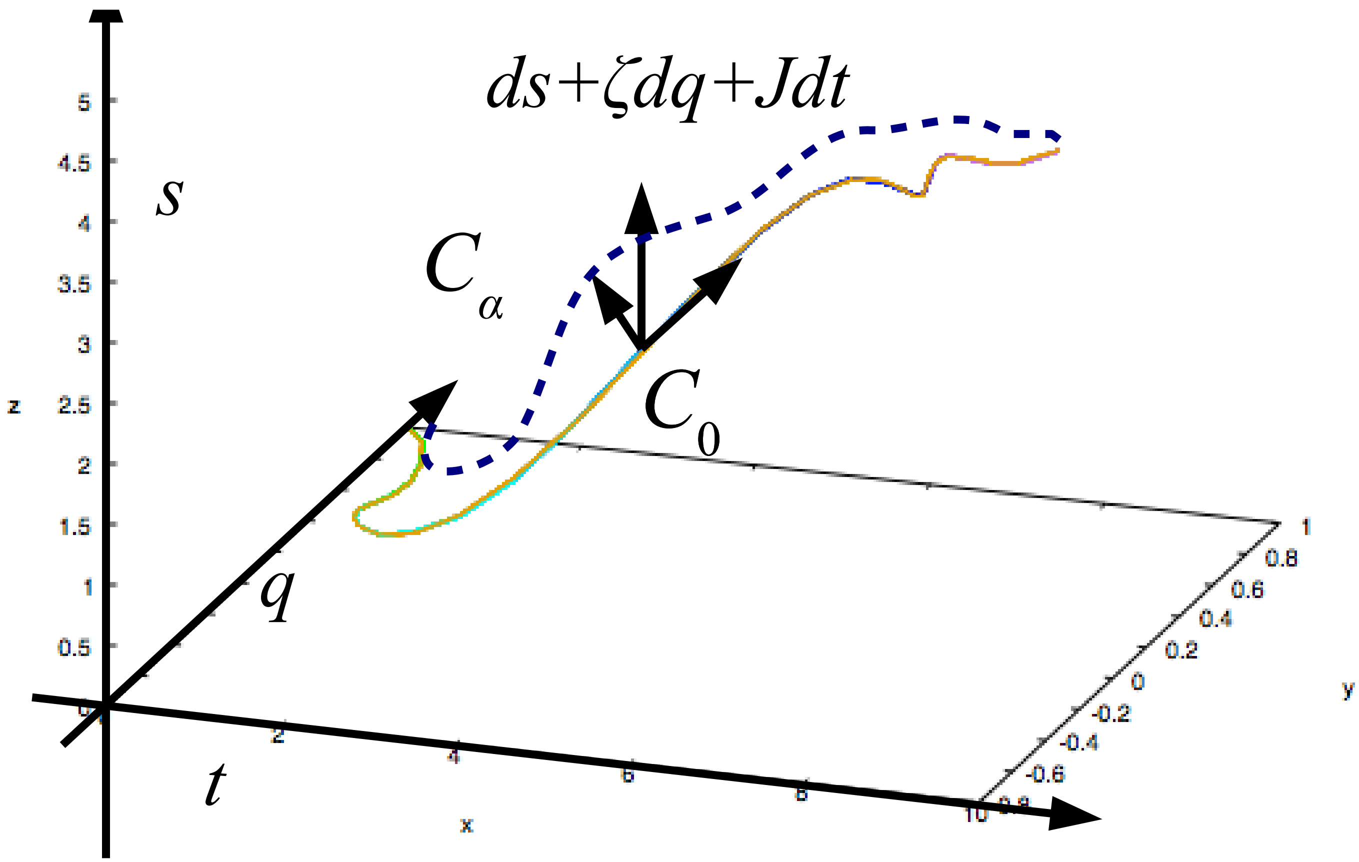
$$\frac{d\Lambda}{dt} = -\frac{\partial H}{\partial s} = -T.$$

A dissipative system
(A non−holonomic constraint)



No surface on which the curves lie.

$ds+\zeta dq+Jdt$ is orthogonal to the tangent vector of C_0 and the virtual displacement $\boldsymbol{X} = \alpha(\delta t, \delta q, \delta s)$.



$$\int_{C_0}\left[\left(\frac{dq}{dt}-\frac{\partial\tilde{H}}{\partial p}\right)\delta p-\left(\frac{dp}{dt}+\frac{\partial\tilde{H}}{\partial q}\right)\delta q-\frac{\partial\tilde{H}}{\partial s}\delta s-\frac{\partial\tilde{H}}{\partial u}\delta u+\frac{d\tilde{H}}{dt}\delta t\right]dt=0$$

$$+)\quad\int_{C_0}T\left(\delta s+\zeta\delta q+J\delta t\right)dt=0$$

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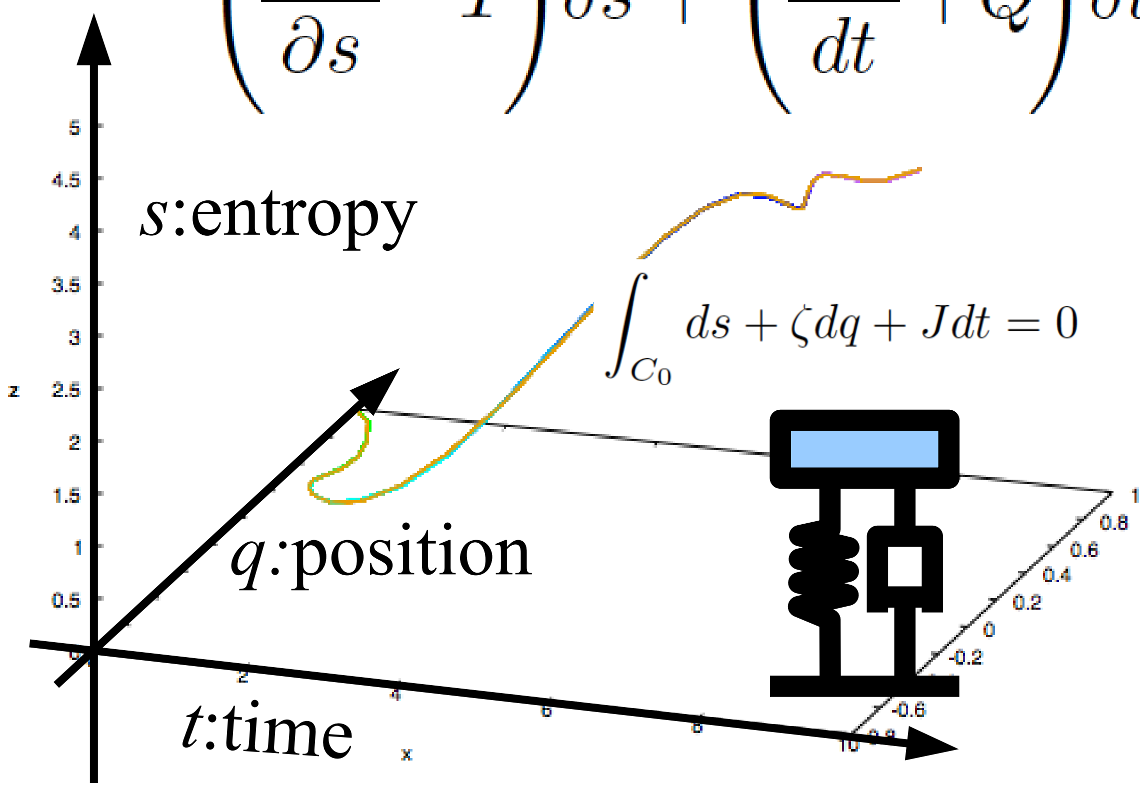
where $T=\dfrac{\partial\tilde{H}}{\partial s}$, $f\equiv T\zeta$, and $Q\equiv TJ$.

$$\int_{C_0}\left[\left(\frac{dq}{dt}-\frac{\partial\tilde{H}}{\partial p}\right)\delta p-\left(\frac{dp}{dt}+\frac{\partial\tilde{H}}{\partial q}-f\right)\delta q-\frac{\partial\tilde{H}}{\partial u}\delta u\right.$$
$$\left.-\left(\frac{\partial\tilde{H}}{\partial s}-T\right)\delta s+\left(\frac{d\tilde{H}}{dt}+Q\right)\delta t\right]dt=0,$$

A damped oscillator

$$\left(\frac{dq}{dt}-\frac{\partial \tilde{H}}{\partial p}\right)\delta p-\left(\frac{dp}{dt}+\frac{\partial \tilde{H}}{\partial q}-f\right)\delta q-\frac{\partial \tilde{H}}{\partial u}\delta u$$

$$-\left(\frac{\partial \tilde{H}}{\partial s}-T\right)\delta s+\left(\frac{d\tilde{H}}{dt}+Q\right)\delta t=0,$$



$$\tilde{H}(p,q,u)=pu-\left(\frac{1}{2}u^2-\frac{1}{2}q^2\right)$$

$$H(p,q)=\frac{1}{2}p^2+\frac{1}{2}q^2$$

Equations of motion:

$$\frac{dq}{dt}=p \qquad \frac{dp}{dt}=-q-f$$

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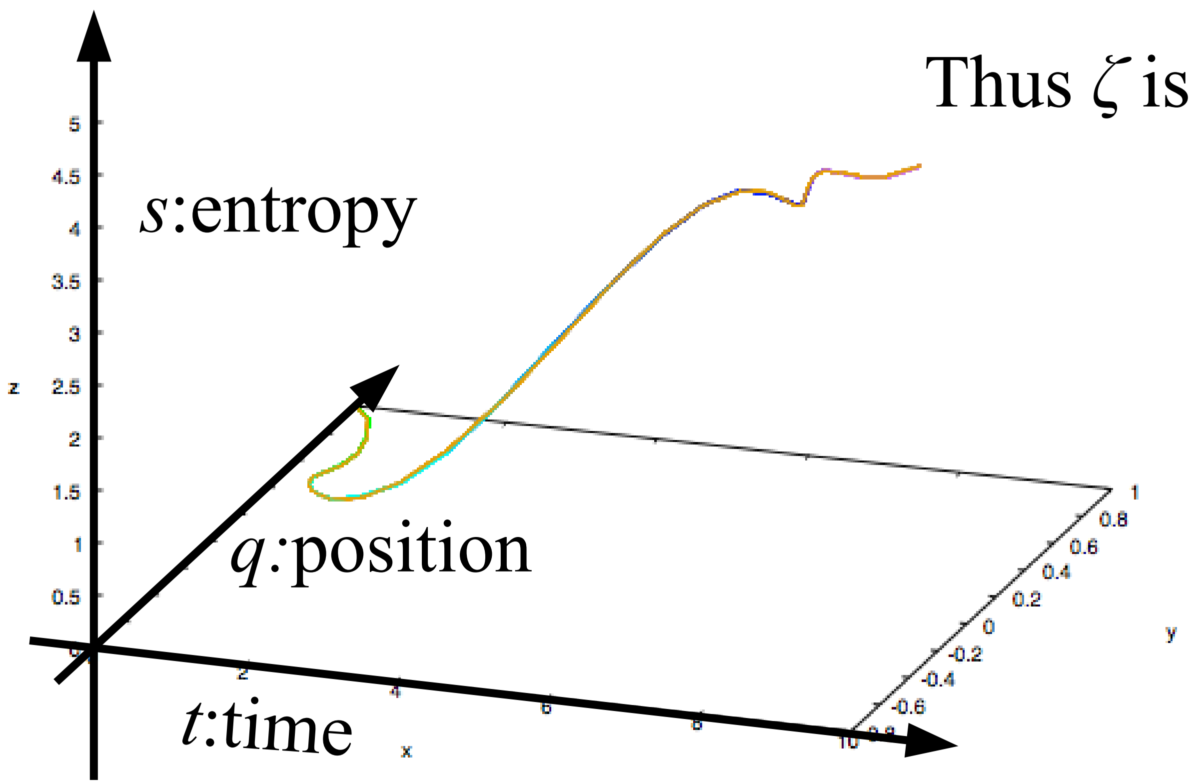
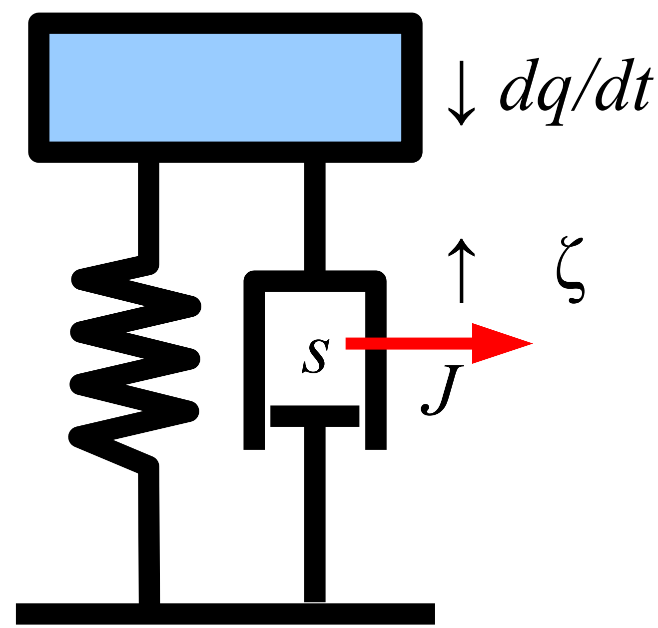
Entropy production

$$\int_{C_0} ds + \zeta dq + J dt = 0 \iff \frac{ds}{dt} = -\zeta \frac{dq}{dt} - J$$

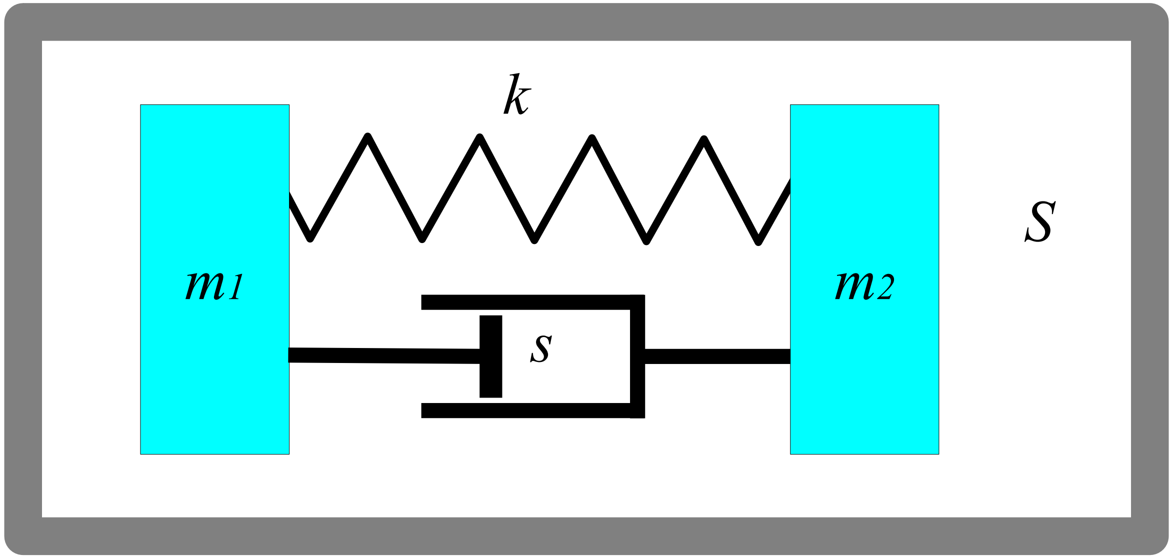
$-\zeta \frac{dq}{dt} \searrow 0$

Thus ζ is opposite sign of dq/dt.





Damped Oscillators



Lagrangian

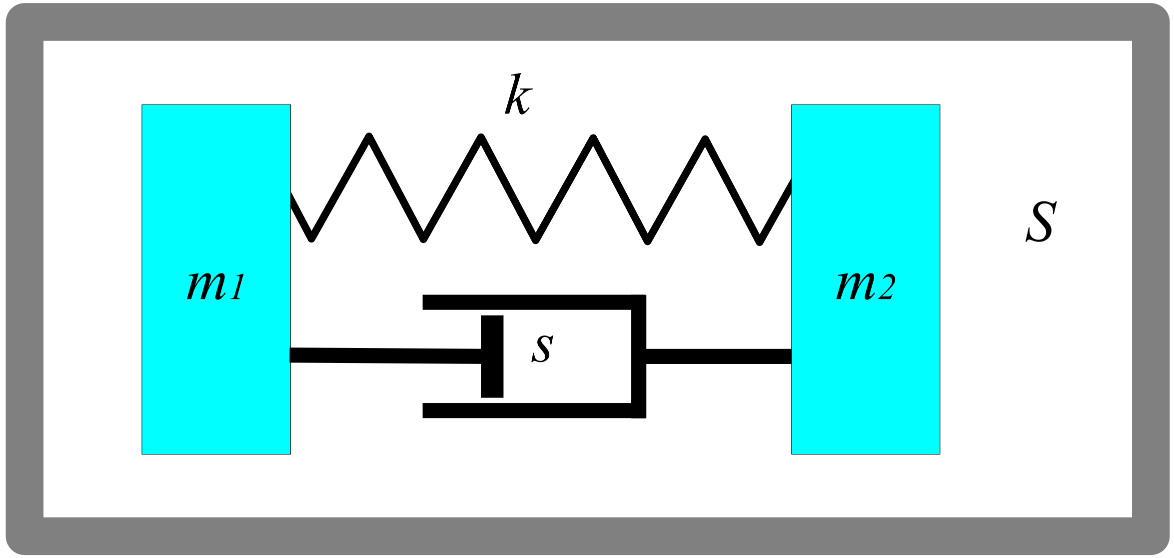
$$\frac{m_i}{2}u_i^2 - \left(\frac{k}{2}(q_1 - q_2)^2 + \epsilon(s) + E(S)\right) + p_i\left(\frac{dq_i}{dt} - u_i\right)$$

Constraint

$$T ds + f_i dq_i + Q dt$$

$$T_E dS + Q_E dt$$

Damped Oscillators



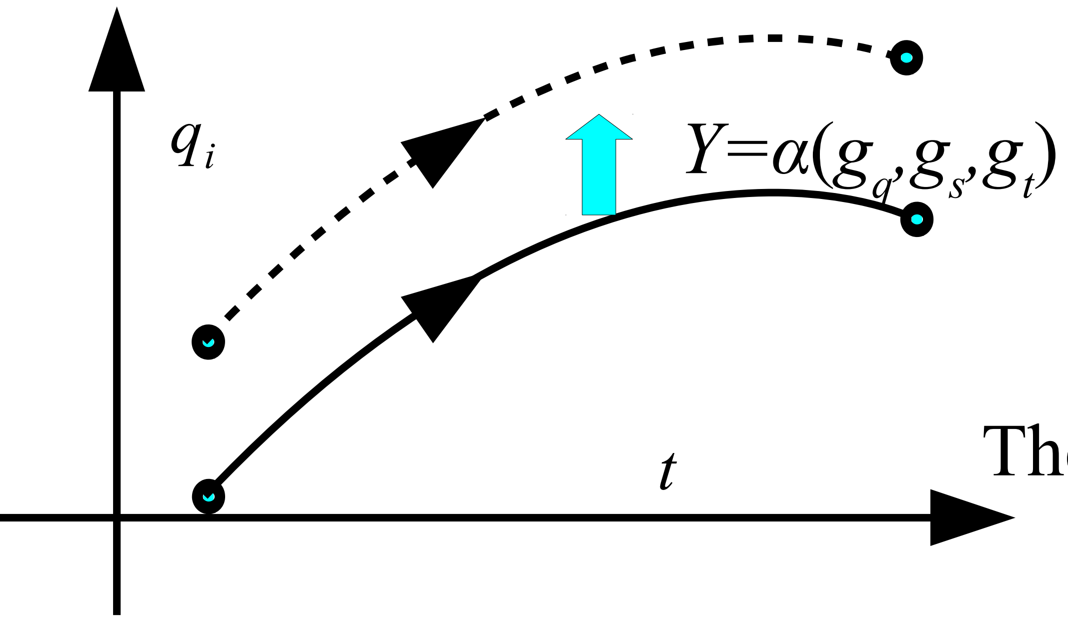
$$
\begin{aligned}
0 &= \frac{\partial \tilde{H}}{\partial u_i}, \\
\frac{dq_i}{dt} &= \frac{\partial \tilde{H}}{\partial p_i}, \\
\frac{dp_i}{dt} &= -\frac{\partial \tilde{H}}{\partial q_i} + f_i.
\end{aligned}
$$

$$m_i u_i = p_i$$

$$\frac{dq_i}{dt} = p_i$$

$$\frac{dp_i}{dt} = \mp k(q_1 - q_2) + f_i$$

Space translation symmetry



$$\boldsymbol{g_q} = (1, \cdots 1)$$
$$g_s = g_t = 0$$

The sufficient condition of

$$L_Y\left(Tds + f_i dq_i + Q\,dt\right) = 0$$

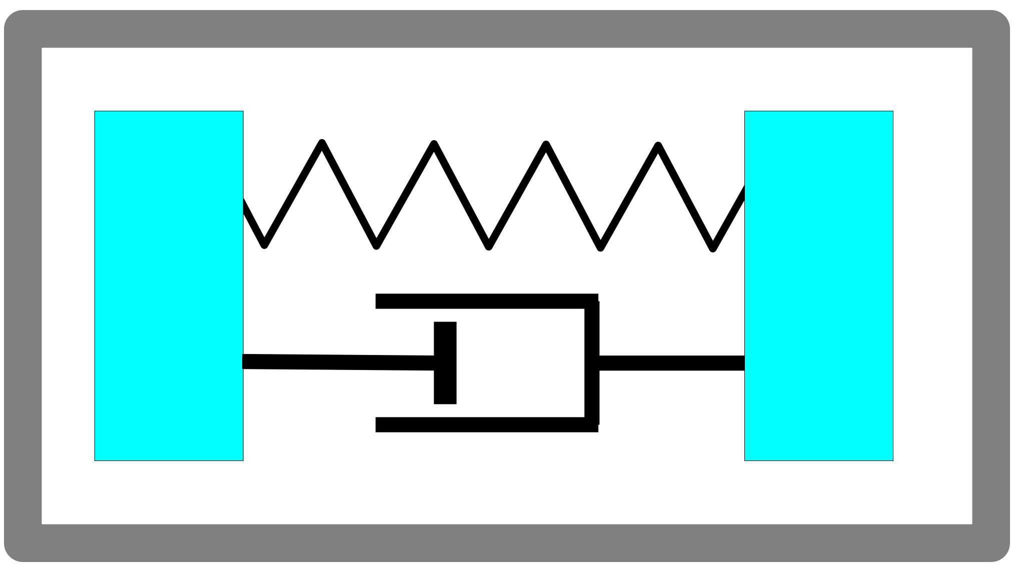
where $L_Y = \left(d\,\iota_Y + \iota_Y\,d\right)$

is

$$Tg_s + \boldsymbol{f}\cdot\boldsymbol{g_q} + Qg_t = 0$$

Thus,

$$\sum_i f_i = 0$$



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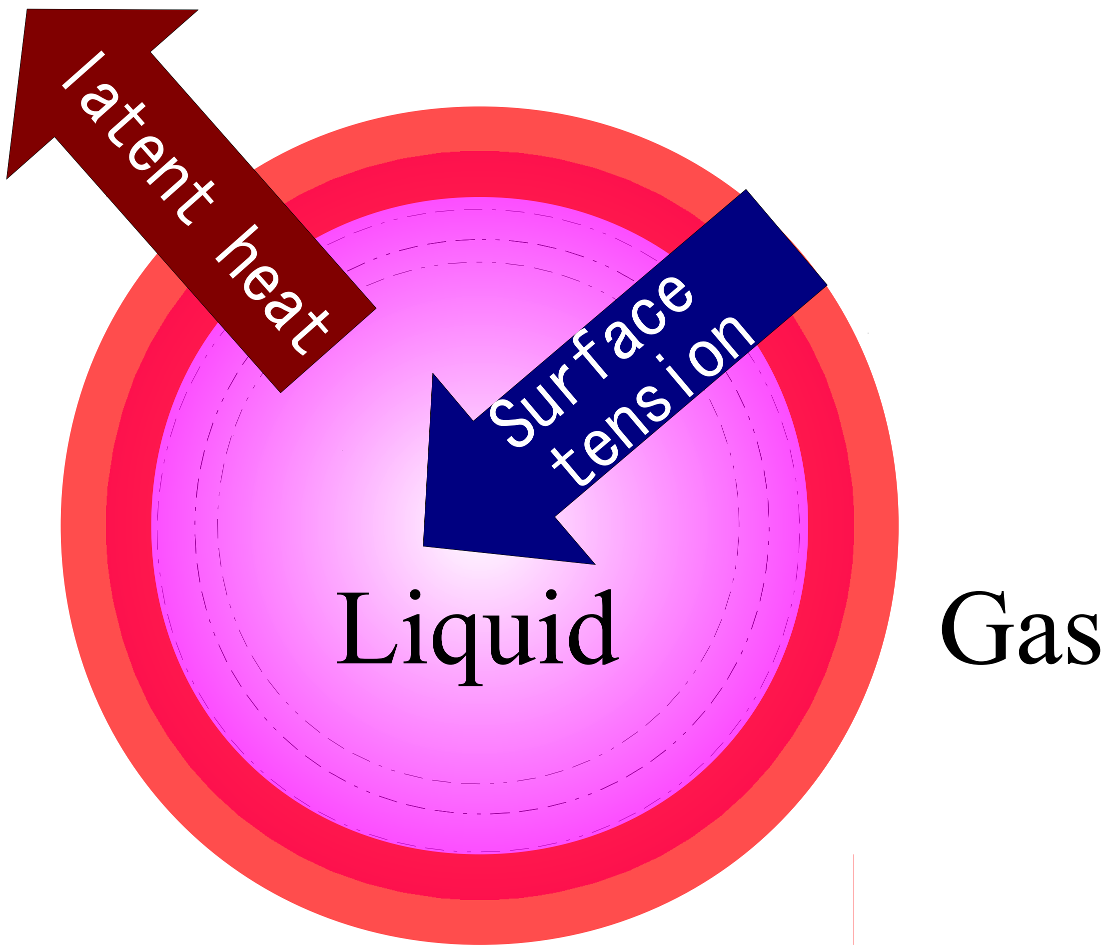
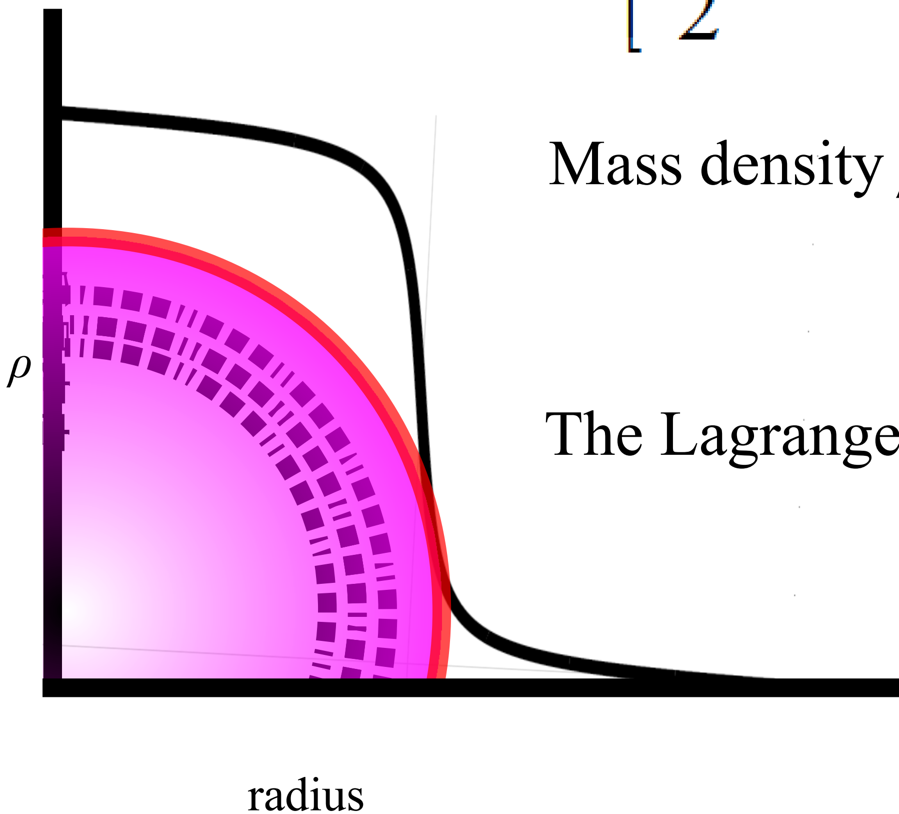
Evaporation



Evaporation

Surface energy

$$\mathcal{L}(\rho, s, \boldsymbol{u}) = \rho\left[\frac{1}{2}u^i u_i - \epsilon(\rho, s)\right] - E(\rho, \nabla\rho)$$



Mass density ρ is a function of the initial position $\boldsymbol{q}$.

$$\rho = \rho_0(\boldsymbol{q})\,\partial(q^1, q^2, q^3)/\partial(x^1, x^2, x^3)$$

The Lagrange derivative of the initial position $\boldsymbol{q}$ is zero.

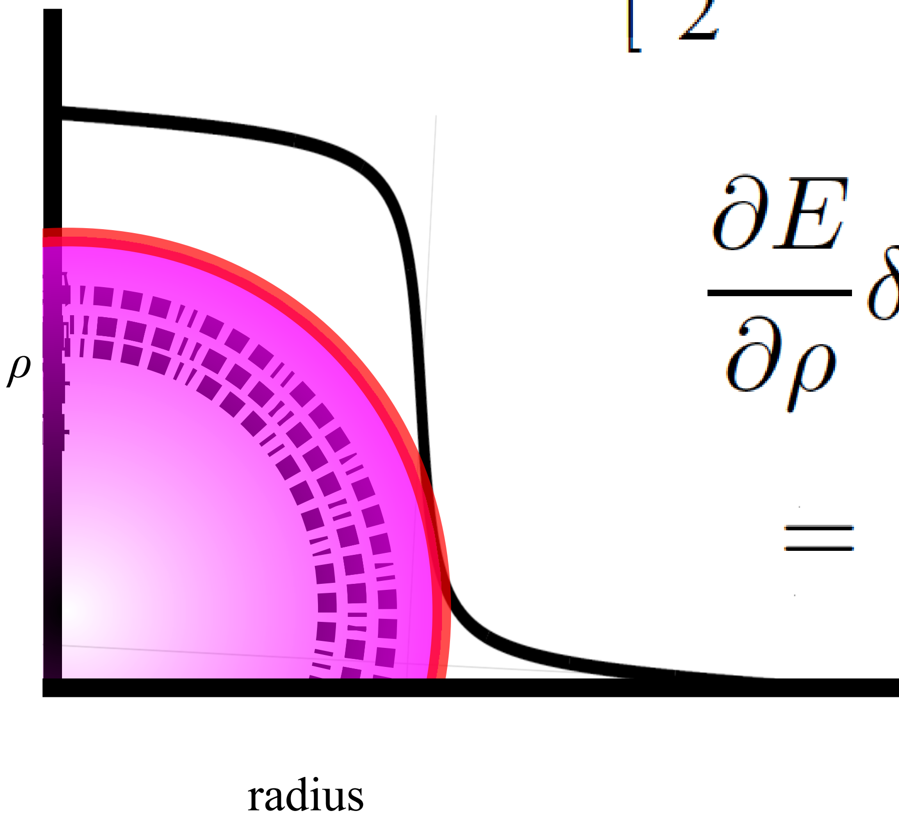
$$D_t q^i \equiv \frac{\partial q^i}{\partial t} + u^j \frac{\partial q^i}{\partial x^j} = 0$$

Then we have $$\delta\rho - \frac{\partial}{\partial x^i}\left(\rho \frac{\partial x^i}{\partial q^j}\delta q^j\right) = 0$$

Evaporation

Surface energy

$$\mathcal{L}(\rho, s, \boldsymbol{u}) = \rho\left[\frac{1}{2}u^i u_i - \epsilon(\rho, s)\right] - E(\rho, \nabla\rho)$$



$$\frac{\partial E}{\partial \rho}\delta\rho + \frac{\partial E}{\partial \nabla\rho}\cdot\delta(\nabla\rho)$$
$$= \left\{\frac{\partial E}{\partial \rho} - \nabla\cdot\left(\frac{\partial E}{\partial \nabla\rho}\right)\right\}\delta\rho$$
$$+ \nabla\cdot\left(\frac{\partial E}{\partial \nabla\rho}\delta\rho\right)$$

Taking the variation of $\int_{t_0}^{t_1} \mathrm{d}t \int_V \mathrm{d}^3\boldsymbol{x} \left[\mathcal{L} + p_i \left(\frac{\partial q^i}{\partial t} + u^j \frac{\partial q^i}{\partial x^j} \right) \right]$

where $\mathcal{L}(\rho, s, \boldsymbol{u}) = \rho \left[\frac{1}{2} u^i u_i - \epsilon(\rho, s) \right]$ - $E(\rho, \nabla \rho)$

yields $\delta \int_V \mathrm{d}^3\boldsymbol{x} \left(p_i \frac{\partial q^i}{\partial t} - \tilde{\mathcal{H}} \right)$

$$\tilde{\mathcal{H}} \equiv -p_i u^j \partial_j q^i - \mathcal{L}$$

$$= \int_V \mathrm{d}^3\boldsymbol{x} \left\{ \left(\frac{\partial q^k}{\partial t} - \frac{\partial \tilde{\mathcal{H}}}{\partial p_k} \right) \delta p_k - \left[\frac{\partial p_k}{\partial t} - \frac{\partial}{\partial x^i} \frac{\partial \tilde{\mathcal{H}}}{\partial \partial_i q^k} \right.\right.$$

$$\left. - \left(\rho \frac{\partial}{\partial x^j} \frac{\partial \tilde{\mathcal{H}}}{\partial \rho} - \frac{\partial \sigma^i_j}{\partial x^i} - T\rho \frac{\partial s}{\partial x^j} \right) \frac{\partial x^j}{\partial q^k} \right] \delta q^k$$

$$- \frac{\partial \tilde{\mathcal{H}}}{\partial u^k} \delta u^k - \left(\frac{\partial \tilde{\mathcal{H}}}{\partial s} - \rho T \right) \delta s + \left(\frac{\partial \tilde{\mathcal{H}}}{\partial t} + \frac{\partial J^i_Q}{\partial x^i} \right) \delta t$$

$$\left. - \frac{\partial}{\partial x^i} \left[\left(\frac{\partial \tilde{\mathcal{H}}}{\partial \partial_i q^k} + \left(\rho \frac{\partial \tilde{\mathcal{H}}}{\partial \rho} \delta^i_j - \sigma^i_j \right) \frac{\partial x^j}{\partial q^k} \right) \delta q^k \right] \right\}.$$

The surface term imposes an extra boundary condition.

$$+ \left\{ \frac{\partial E}{\partial \rho} - \nabla \cdot \left(\frac{\partial E}{\partial \nabla \rho} \right) \right\} \delta \rho + \nabla \cdot \left(\frac{\partial E}{\partial \nabla \rho} \delta \rho \right)$$

$$\int_V d^3\boldsymbol{x}\left\{\left[\left(\frac{\partial q^k}{\partial t}-\frac{\partial\tilde{\mathcal{H}}}{\partial p_k}\right)\delta p_k-\left[\frac{\partial p_k}{\partial t}-\frac{\partial}{\partial x^i}\frac{\partial\tilde{\mathcal{H}}}{\partial\partial_i q^k}\right.\right.\right.$$

$$\left.-\left(\rho\frac{\partial}{\partial x^j}\frac{\partial\tilde{\mathcal{H}}}{\partial\rho}-\frac{\partial\sigma^i_j}{\partial x^i}-T\rho\frac{\partial s}{\partial x^j}\right)\frac{\partial x^j}{\partial q^k}\right]\delta q^k$$

$$-\frac{\partial\tilde{\mathcal{H}}}{\partial u^k}\delta u^k-\left(\frac{\partial\tilde{\mathcal{H}}}{\partial s}-\rho T\right)\delta s+\left(\frac{\partial\tilde{\mathcal{H}}}{\partial t}+\frac{\partial J^i_Q}{\partial x^i}\right)\delta t$$

$$\left.\left.-\frac{\partial}{\partial x^i}\left[\left(\frac{\partial\tilde{\mathcal{H}}}{\partial\partial_i q^k}+\left(\rho\frac{\partial\tilde{\mathcal{H}}}{\partial\rho}\delta^i_j-\sigma^i_j\right)\frac{\partial x^j}{\partial q^k}\right)\delta q^k\right]\right\}:$$

Well-posedness

Mathematical models of physical phenomena should have the properties that

(1) A solution exists

(2) The solution is unique

(3) The solution depends continuously on the initial conditions and the boundary conditions.

$$+\left\{\frac{\partial E}{\partial\rho}-\nabla\cdot\left(\frac{\partial E}{\partial\nabla\rho}\right)\right\}\delta\rho \quad +\nabla\cdot\left(\frac{\partial E}{\partial\nabla\rho}\delta\rho\right)$$

In order to have a solution, eliminate the extra surface term by the non-holonomic constraint

$$\rho D_t s=\frac{1}{T}\left(\sigma^i_j e^j_i-\nabla\cdot\boldsymbol{J}_q\right)-\nabla\cdot\boldsymbol{J}_s$$

where

$$e^i_j\equiv\delta^{ik}\left(\partial_j u_k+\partial_k u_j\right) \qquad \boldsymbol{J}_s=\frac{1}{T}\frac{\partial E}{\partial\nabla\rho}D_t\rho$$

Equations of motion

$$\partial_t(\rho u_i)+\partial_j(\rho u^j u_i+\Pi_i^j+\sigma_i^j)-\gamma_i=0$$

$$\gamma_i \equiv \frac{\partial_j T}{T}\left(\frac{\partial E}{\partial\partial_i\rho}\partial_j\rho-\frac{\partial E}{\partial\partial_j\rho}\partial_i\rho\right)$$

$$\Pi_i^j\equiv\left[P-\rho T\partial_k\left(\frac{1}{T}\frac{\partial E}{\partial\partial_k\rho}\right)+\rho\frac{\partial E}{\partial\rho}-E\right]\delta_i^j+\frac{\partial E}{\partial\partial_j\rho}\partial_i\rho$$

Summary

The dynamics of a physical system can be described by a trajectory in a configuration space. The trajectory is determined by a law of its tangent vector, i.e., the generalized velocity. In terms of a control theory, the velocity is regarded as the control parameter which determines the trajectory. The law of velocity is universalized into a policy of control. Hamilton's principle is generalized as an optimal control theory, which seeks the control giving the stationary value of a cost functional.

In a dissipative system, entropy depends on the time development of other variables in the configuration space. This relation is given by a set of non-holonomic constraints. Then the equations of motion are obtained by solving the stationary condition of a cost functional subject to the non-holonomic constraints.

In this formulation, physical systems are characterized by the sets of a functional and non-holonomic constraints. All are consistent with symmetries and well-posedness. Moreover, the constraint of entropy satisfies the law of entropy increase. These restrictions define the proper class for equations of motion in physics.

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