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PPPL PRINCETON PLASMA PHYSICS LABORATORY

# Putting the D in M℞MHD

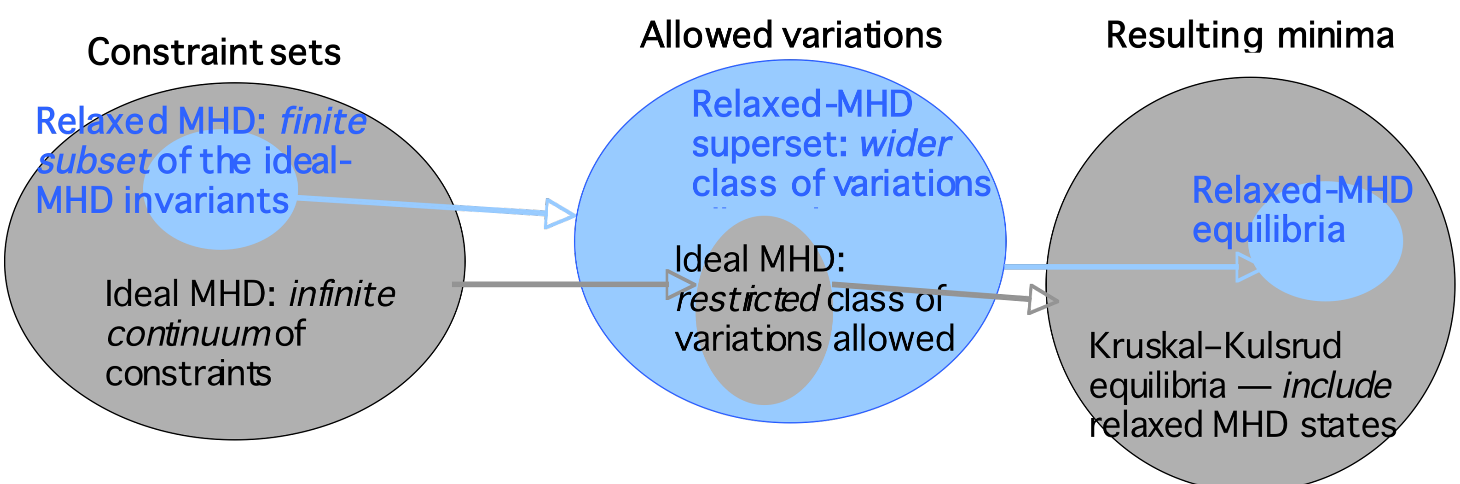
## a prescription for all that ails ideal MHD!

New paper: R.L. Dewar,[1,2] Stuart Hudson,[2]
Amitava Bhattacharjee,[2] Zensho Yoshida[3]
Phys. Plasmas: accepted March 10, 2017!

*1. Plasma Theory & Modelling, Centre for Plasmas & Fluids, RSPE, Australian National University*
*2. Princeton Plasma Physics Laboratory*
*3. Dept. of Advanced Energy, University of Tokyo*

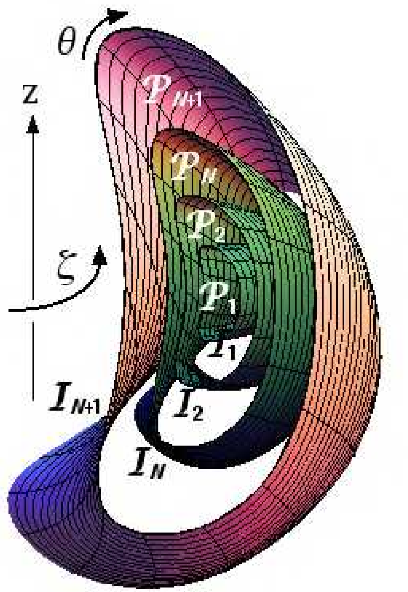
# Why and how to fix Ideal MHD?

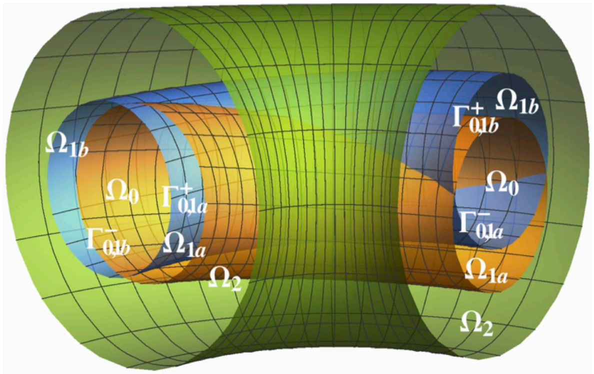
- Ideal MHD is overconstrained
  - No heat transport along field lines
  - No reconnection so islands or chaos cannot form
  - Thus *inapplicable* to hot and 3D plasmas!
- Fix by removing the bad constraints and keeping the good, doing more with less!



# MRxMHD: M stands for **Multi-region** (aka waterbag)

# Rx stands for **Relaxed**; ..D stands for **Dynamics**




## Fundamental postulates of new **general reformulation of MHD:**

- ∃ transport interfaces, $\mathcal{I}_i$ or $\Gamma_{i,j}$, or $\partial\Omega_{i,j}$ (e.g. nested tori or island separatrices), that act like sheets of ideal-MHD plasma
- Plasma relaxes (in some generalized Taylor sense) in regions $\mathcal{P}_i$ (or $\Omega_i$) bounded by the interfaces
- Only a *subset* of ideal-MHD invariants apply

# SPEC (currently) uses MRxMHS, not MRxMHD:

MRxMHS = Multi-region Relaxed Magnetohydro*Statics* (i.e. equilibrium theory)

Ref. Stuart Hudson's talk yesterday

- Taylor relaxation *energy* principle
- constant *pressure* in each region

MRxMHD = Multi-region Relaxed Magnetohydro*Dynamics*

*New* approach:: use *Hamilton's Principle* — stationarity of time-integrated *Lagrangian*

➩ constant *temperature* in each region

➩ supports sound waves within relaxation regions as well as radially compressible and Alfvén modes + *tearing*

➩ can treat *development* of resonant current sheets

➩ can add equilibrium flow to SPEC and will be basis for a new time-evolution waterbag code

# MRxMHD Lagrangian is *kinetic energy* minus MHD *potential energy* + constraint terms:

- MHD Lagrangian density in region $i$

$$\mathcal{L}^{\mathrm{MHD}} = \rho\frac{v^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$$

- Constrained Lagrangian in region $i$

$$L_i = \int_{\Omega_i} \mathcal{L}^{\mathrm{MHD}} dV + \tau_i(S_i - S_{i0}) + \mu_i\left(K_i - K_{i0}\right)$$

- Helicity and entropy *macroscopic* invariants

$$K_i \equiv \int_{\Omega_i} \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0} dV \qquad S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln\left(\kappa\frac{p}{\rho^\gamma}\right) dV$$

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# In varying action, $\rho$ is constrained *holonomically* to the displacement $\xi$ of each fluid element:

- Mass conserved *micro*scopically, i.e. pointwise

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}) \text{ in } \Omega_i$$

- Helicity and entropy constrained *macro*scopically, throughout $\Omega_i$, using Lagrange multipliers $\mu_i$ and $\tau_i$, while *p and* **A** are free fields
- Including vacuum field energy, total Lagrangian is

$$L = \sum_i L_i - \int_{\Omega_\mathrm{v}} \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}\, dV$$

- Setting variation of action to 0 gives EL equations:

$$\delta \int L dt = 0$$

# Equations within $\Omega_i$

- Mass conservation (microscopic constraint)

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho \mathbf{v})$$

- $\delta p$ ➪ Isothermal equation of state

$$p = \tau_i \rho \quad (\text{N.B. } \tau_i = C_{\text{s}i}^2)$$

- $\delta\mathbf{A}$ ➪ Beltrami equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_i \mathbf{B} \quad (\text{N.B. } \Rightarrow \mathbf{j}\times\mathbf{B} = 0)$$

- $\boldsymbol{\xi}$ ➪ Momentum equation (Euler fluid)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla} p$$

# Equations on interface $\Gamma_{i,j}$

- $\boldsymbol{\xi}$ ⇨ Force balance

$$\left[\!\left[ p + \frac{B^2}{2\mu_0} \right]\!\right]_{i,j} = 0$$

- Surface constraints

$$\mathbf{n}_i \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega_i$$

$$\mathbf{n}_i \cdot [\![\mathbf{v}]\!]_{i,j} = 0 \quad \text{on } \partial\Omega_{i,j}$$

- Complete set of equations, consistent because derived from single scalar function *L*

# Proving the MRxMHD pudding:

- *Q1) What is the MRxMHD spectrum and what are the effects of field-line curvature and equilibrium mass flow on stability?*
- *Q2) When are the current sheets topologically stable towards internal plasmoid formation (reconnection)?*
- *Q3) When do unstable modes saturate at a low level or develop nonlinearly into explosive events?*

# What happens in static or adiabatic limit?

- $\partial_t \to 0$ ➪ $\nabla\cdot(\rho\mathbf{v}) = 0 \qquad \mathbf{v}\cdot\nabla\mathbf{v} = -\tau_i \nabla \ln\rho$

➪ only solutions valid for *any* flowline configuration, from nested surfaces to arbitrarily chaotic, are

$$\rho = \rho_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right) \Rightarrow p = p_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right)$$

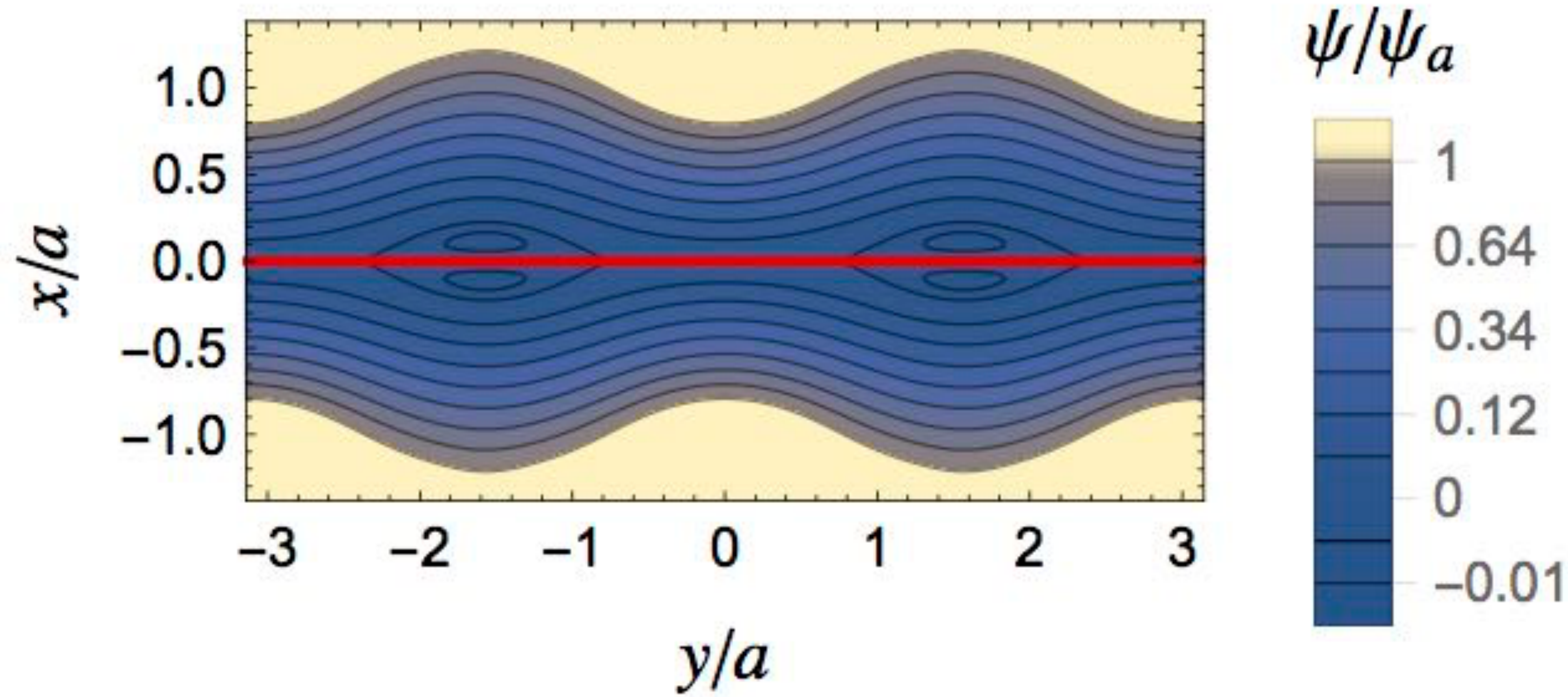
(N.B. *incompressible* in limit $v/C_\mathrm{s} \to 0$) and

$$\nabla\times\mathbf{v} = \alpha_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right)\mathbf{v}$$

Almost isomorphous to **B** equation: should be implementable in SPEC. Derivable variationally — N. Sato

# Slow (adiabatic) limit

## Switch on slab boundary ripple to study Resonant Magnetic Perturbations (RMPs)




MRxMHD Hahm-Kulsrud-Taylor (HKT):
Rippled Slab Model for resonant current sheets

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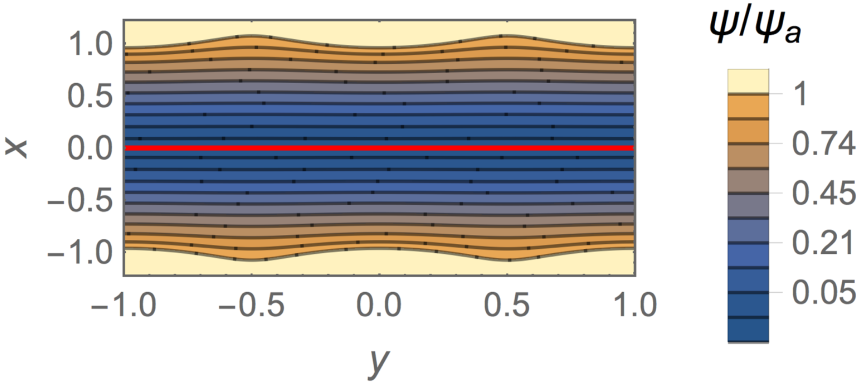
# 2-region MRxMHD model*




*From APS DPP 2014 poster, with Finn-Antonsen helicity

- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$

Hahm & Kulsrud (HK), Phys. Fluids '85, found 2 solutions:

- shielding current sheet on $x = 0$ (shown in red)

$$\psi = aB_y^a \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

$$\psi = aB_y^a \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where $B^a{}_y$ is |unperturbed poloidal field| at boundaries and $\alpha \ll 1$

# A good test case for MRxMHD:

- Linearity of Beltrami equation leads to easily solvable, *linear* GS equation *(Poisson in small-$\mu$ limit.)*
- Symmetry about, and straightness of, current sheet at $x = 0$: gives most
- geometrically simple *2-region relaxation scenario:*

- Switch-on: *ripple* on upper and lower boundaries slowly increased from zero (plane slab) to final amplitude
- A *shielding current* sheet at $x = 0$ resonance develops
- Kruskal-Kulsrud damping: evolution through *equilibria*
- Connect equilibrium sequence by *helicity conservation*

# Grad-Shafranov-Beltrami equations

Grad-Shafranov equation for force-free field in slab geometry:

$$\mathbf{B} = \nabla z \times \nabla \psi + F(\psi)\nabla z \qquad \nabla^2\psi + FF' = 0$$

$\nabla \times \mathbf{B} = \mu \mathbf{B}$ (Beltrami equation) is satisfied by requiring:

$\nabla^2\psi = \mu F$ with $F(\psi) = C - \mu\psi$ , giving $(\nabla^2 + \mu^2)\psi = C$

General Solution: $\psi = \overline{\psi} + \dfrac{\overline{F}}{B_0}\psi_0(x|\mu) + \widehat{\psi}(x,y)$

where $\overline{\psi}$ is cross-sectional average of $\psi$,

$$\psi_0(x|\mu) \equiv \frac{B_0}{\mu}(1 - \cos \mu x)$$

is plane slab solution, $\overline{F}$ is the cross-sectional average of $B_z$,

and $\widehat{\psi}$ obeys a *homogeneous* Beltrami equation $(\nabla^2 + \mu^2)\widehat{\psi} = 0$

with boundary conditions such that $\psi$ is constant on boundary and on cuts.

# Extension of HK shielding solution

*Helicity conservation requires three extensions of HK solution: Instead of the HK harmonic component  $\psi_1$ we use ansatz*

$$\widehat{\psi}(x,y) \equiv \frac{2\alpha\psi_a}{\sinh k_1 a}\left(|\sinh k_1 x|\cos ky + \gamma_\mathrm{S}\frac{k_1}{\mu}|\sin\mu x|\right) - \overline{\psi}\cos\mu x$$

where:

1. $\widehat{\psi}$ is a solution of the *Beltrami equation* $(\nabla^2 + \mu^2)\widehat{\psi} = 0$ It is only *harmonic* in the *small-μ limit.* Likewise

   $$k_1(\mu) \equiv (k^2 - \mu^2)^{1/2} \to k \text{ only as } \mu \to 0$$

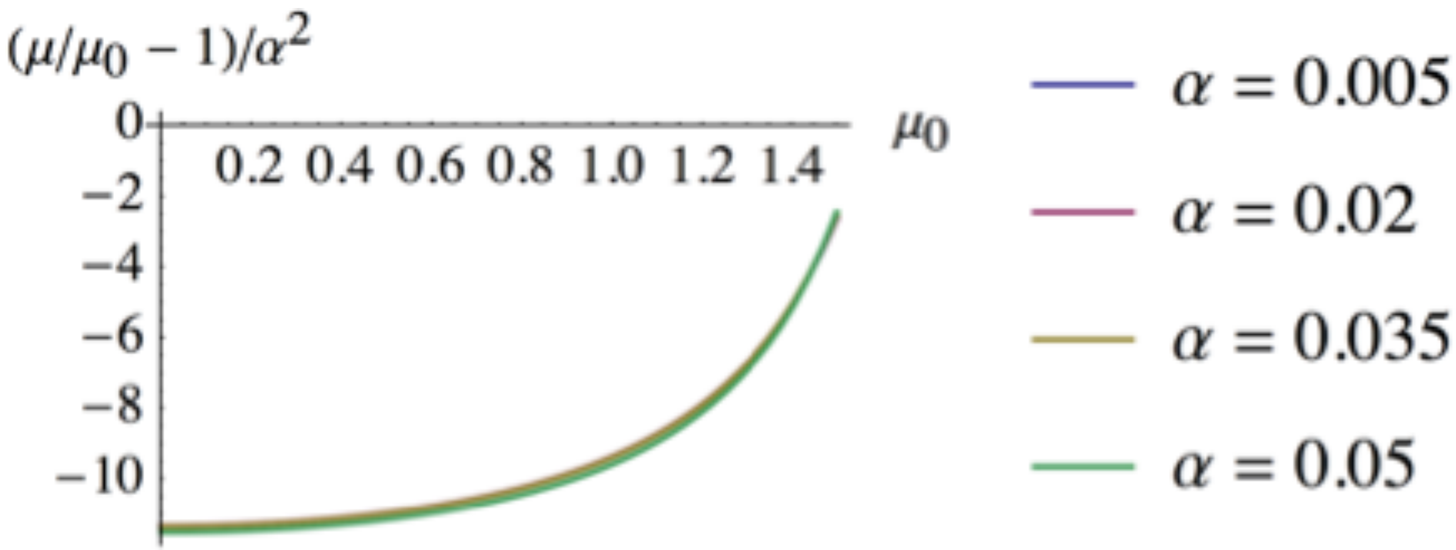
2. The term in $\gamma_\mathrm{S}$ was introduced in Dewar *et al*. 2013 to allow control of the *total current* in the sheet
3. The term in $\overline{\psi}$ is required for poloidal flux conservation

# $\mu$ is not fixed

- In plane slab, *before* ripple is turned on, the *unperturbed* equilibrium flux function is

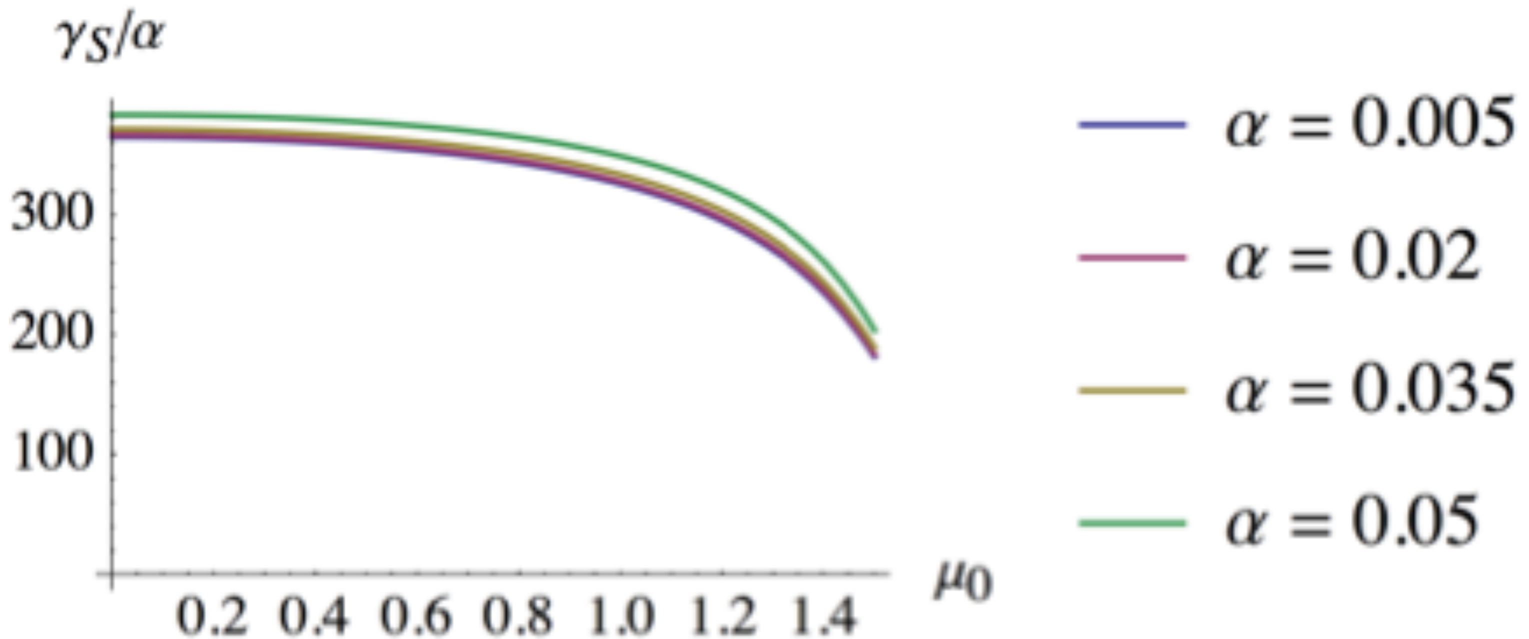
$$\psi_0(x|\mu_0) \equiv \frac{B_0}{\mu_0}(1 - \cos \mu_0 x)$$

- As amplitude parameter $\alpha$ is increased from 0, $\mu$ must *change* to preserve helicity and fluxes:

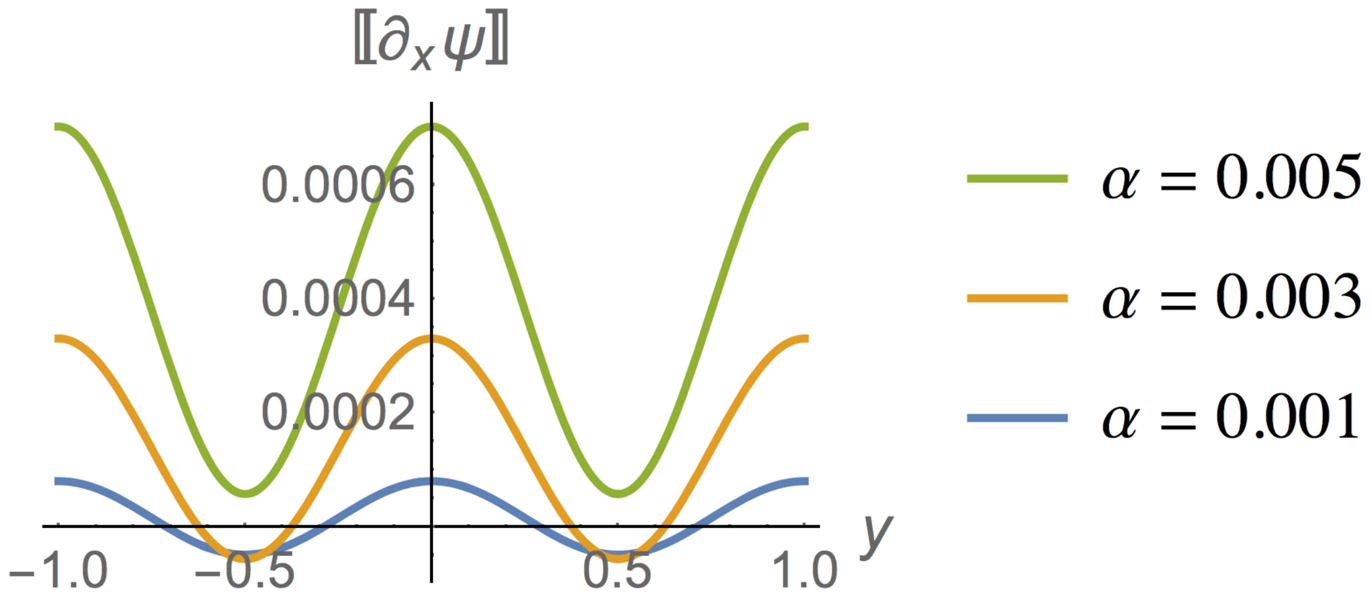


# Current sheet has a strong d.c. component

- HK implicitly assumed the total current in the sheet was zero, but MRxMHD switch-on shows there is a *nonzero* total current $J = \frac{2\alpha\psi_a k_1 \lambda}{\sinh k_1 a} \gamma_S$ proportional to $\gamma_S$ :

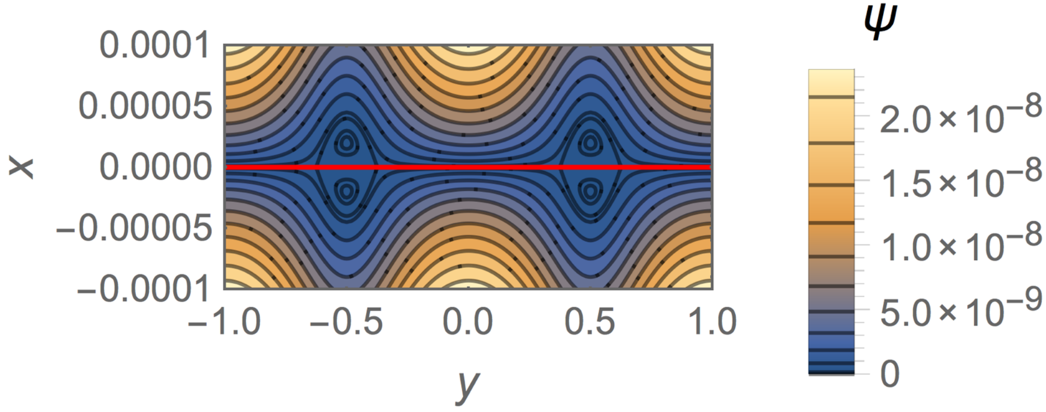


# Current sheet reverses for small perturbations



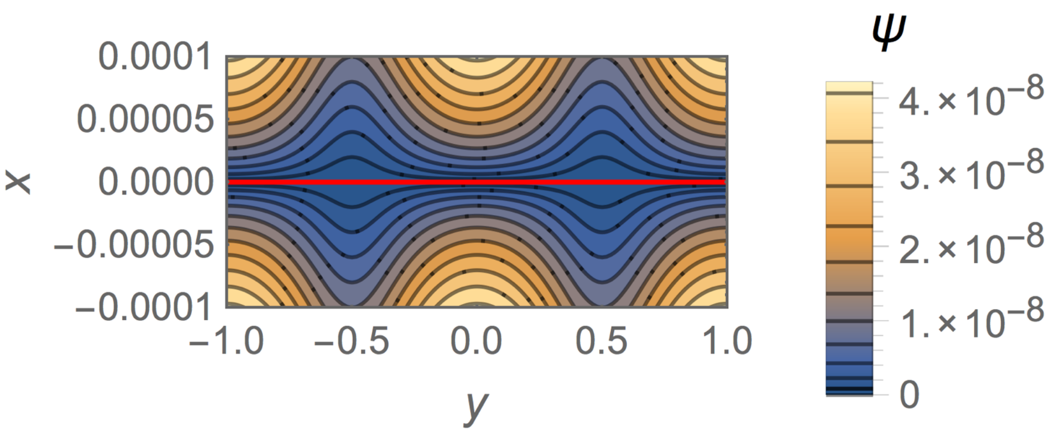

Fully shielded case: Plots of the jump in the gradient of $\psi$, vs. $y$ for $\mu_0$ = 1.4 and selected small values of $\alpha$, showing the occurrence of current-density reversal for the two smallest values.

# Current reversals cause ½ islands!




Ripple amplitude:
$\alpha = 0.003$
Current density exhibits sign reversal

2-region MRxMHD Hahm-Kulsrud model: mirror-image ripple top and bottom excites modulated current sheet at $x = 0$




Larger ripple amplitude:
$\alpha = 0.005$
No sign reversal so half-islands disappear

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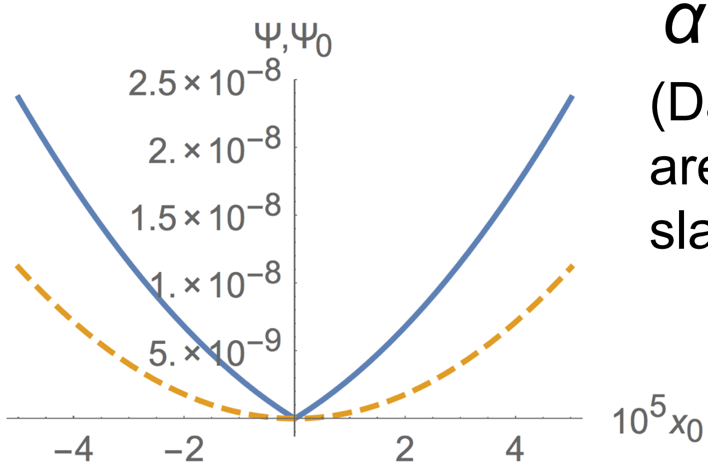
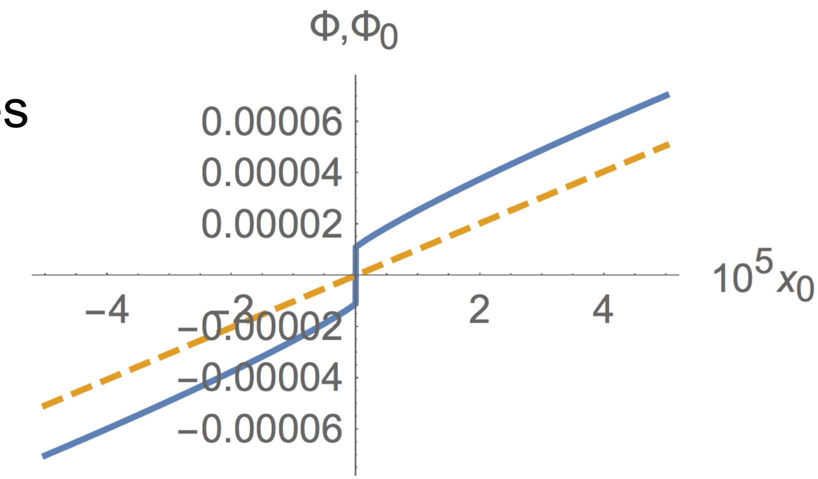
# Fluxes and transform I

**$\alpha$ = 0.001**

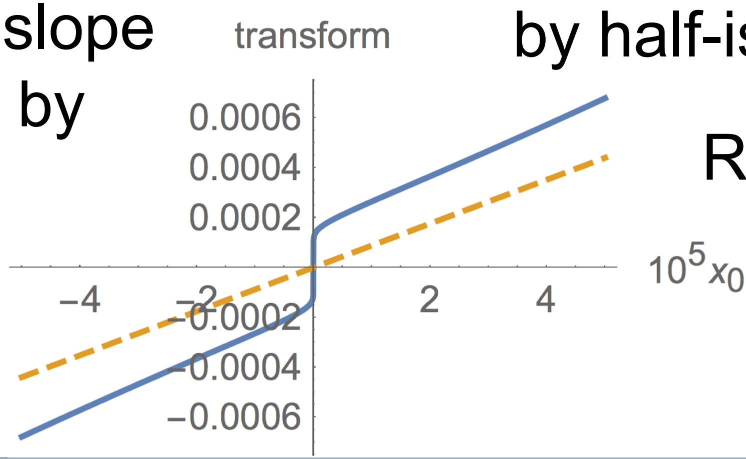
(Dashed curves are for plane slab, α = 0)




Poloidal flux as a function of $x_0$ (= $x$ along $y$-axis), showing discontinuity in slope at $x$ = 0 caused by current sheet




Toroidal flux as a function of $x$ along $y$-axis, showing discontinuity at $x$ = 0 caused by half-island.

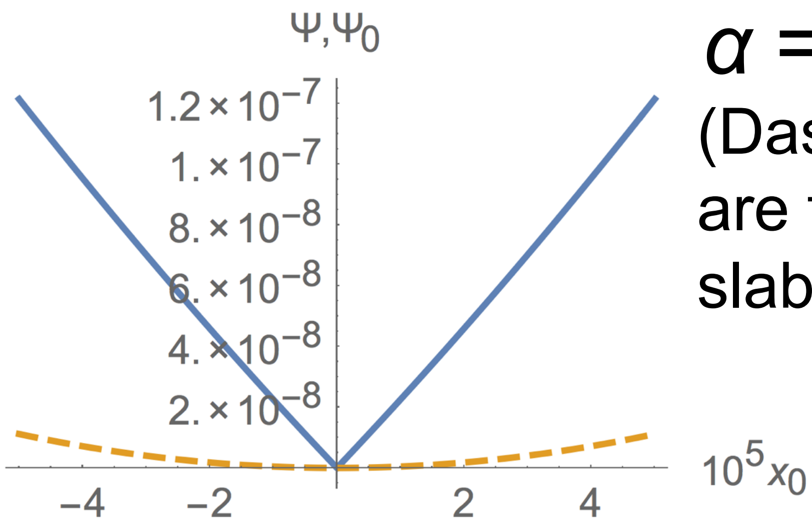



Rotational transform (1/q)

$$\Psi'(x_0)/\Phi'(x_0)$$

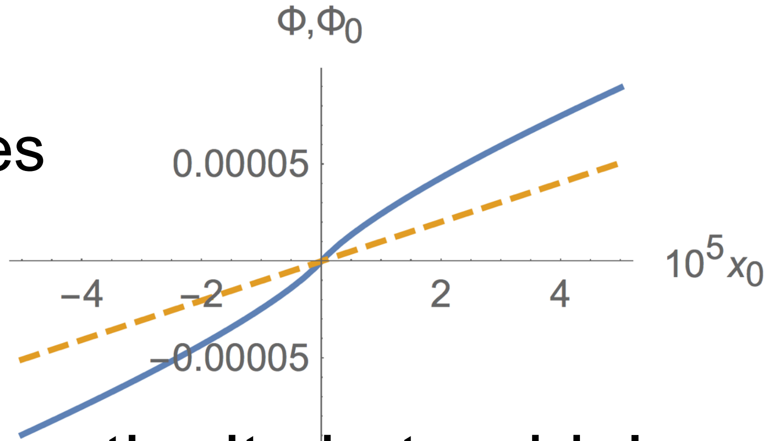
showing jump or large slope near $x_0$ = 0.

# Fluxes and transform II

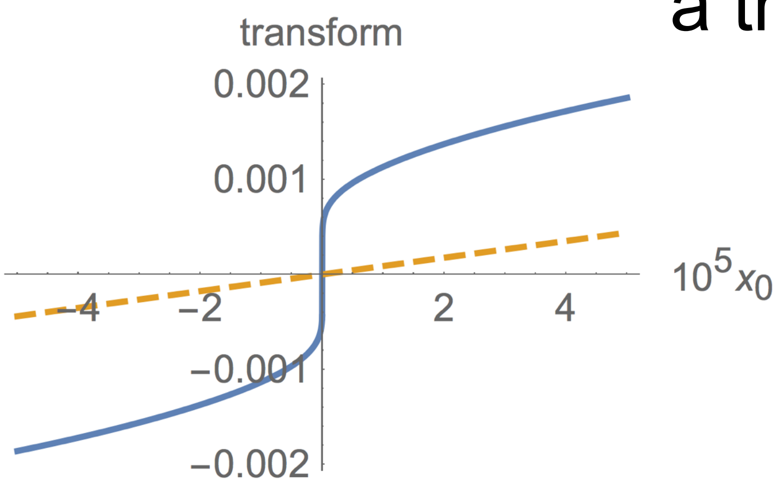



**$\alpha$ = 0.005**

(Dashed curves are for plane slab, α = 0)




Discontinuity in toroidal flux has gone as there are no half-islands above a threshold in α c. 0.0045



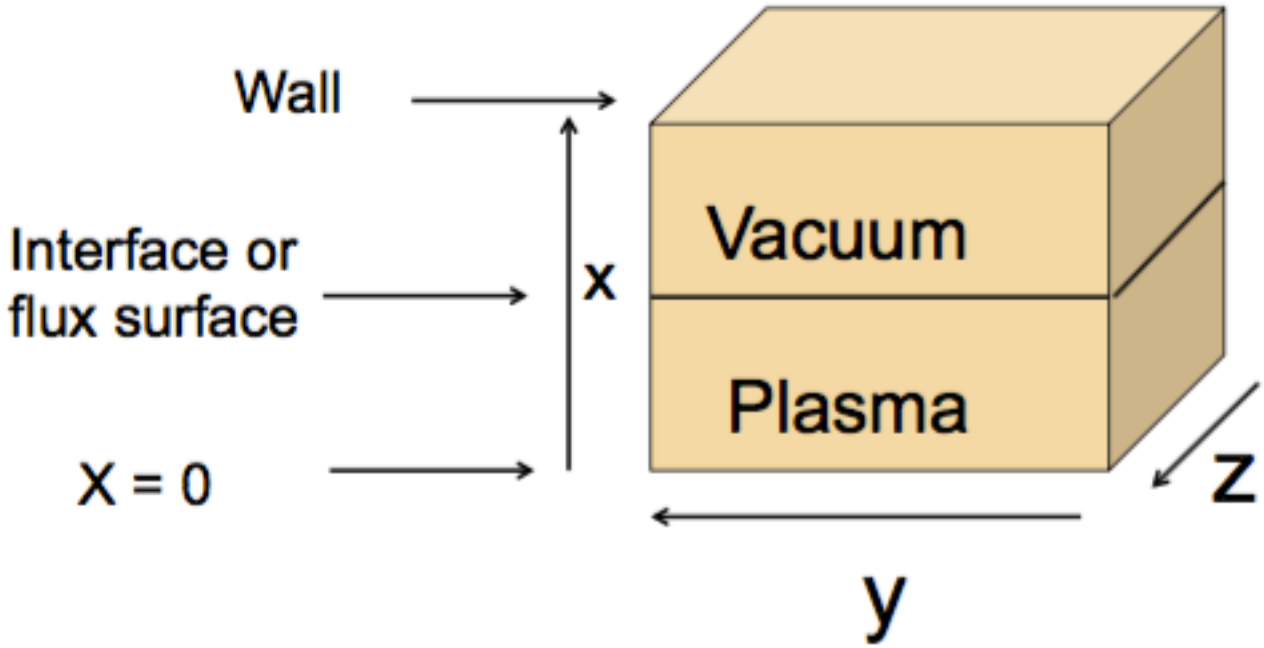

Much stronger jump in rotational transform

# Full *t*-dependence: linear modes in slab

$B_1$ a superposit$^n$ of “Beltrami waves” in plasma ($\mu > 0$)

*and*

vacuum ($\mu = 0$)

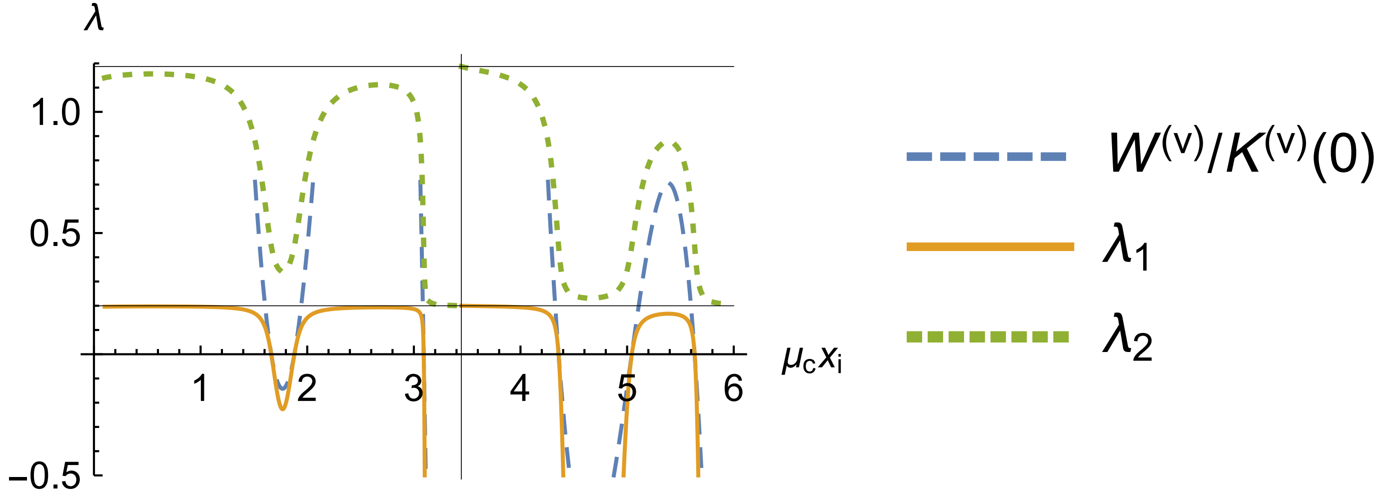



New MRxMHD: sound waves in plasma ($\rho_0$ = const > 0, $\tau > 0$)

Old MRxMHS+: $\lambda = \omega^2$ with $\rho_0 = \delta(x\text{-}a)$ ⇨ no sound waves Hole et al, Nucl Fusion **47**, 746 (2007), etc

Alexis Tuen’s MSc thesis 2016

# First two eigenvalues, + incompressible approximation at very small $\lambda \equiv \omega^2$




- Dewar, Tuen, Hole: Plasma Phys. Control. Fusion **59**, 044009, (2017)
- Growth rate zero if wall or **k.B**=0 is at interface

# Detailed Conclusions

- Multi-region generalization of Taylor relaxation has been extended to a self-consistent dynamics through Hamilton’s Principle of Stationary Action.
- A rippled slab model has been used to illustrate the formation of a resonant current sheet as boundary ripple is switched on
- For very small ripple amplitudes current reversal occurs in the current sheet and unperturbed sheared magnetic field exhibits topological change, with small half-islands, locking rotational transform to resonant value
- For larger ripple amplitude the rotational transform jumps across the current sheet

# General Conclusion

- Action-based MRxMHD shows great promise
  - Very simple
  - Includes reconnection and flow in natural way
- 1st: check physical reasonability of predictions in simple models
- 2nd: Extend SPEC to flow; build new time-evolution and normal mode codes