Nonlinear plasma physics for fusion

finding our way on the green side of the nuclear valley

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Abstract

Negotiating the path to fusion power requires understanding and controlling a very complex system: a plasma sustaining an enormous thermal gradient that drives many emergent nonlinear phenomena through a variety of self-organization mechanisms. After a very brief overview of the magnetic confinement approach to fusion power, some theoretical approaches to understanding and modeling these phenomena will be reviewed.

Plan

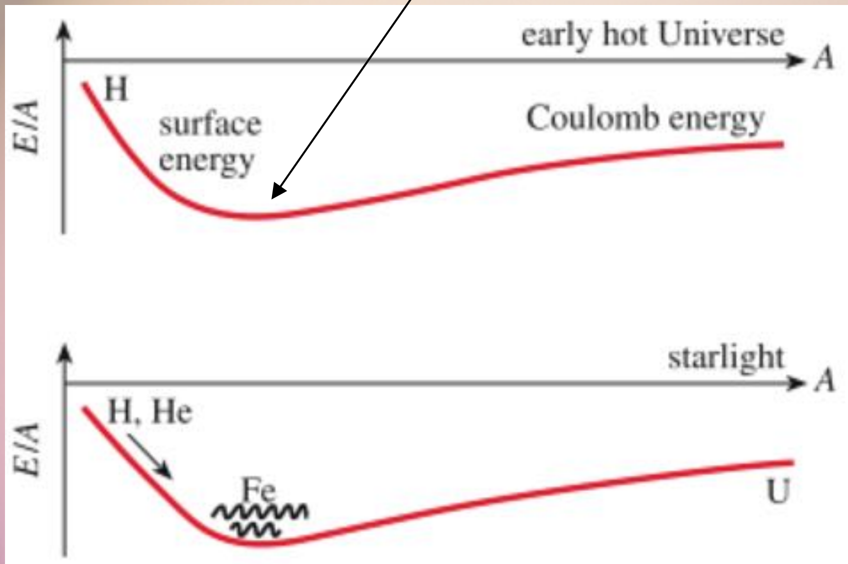
- Part1: Overview of some key concepts
  - Magnetic confinement approach to fusion
    - Particle and magnetic field dynamics
    - Complex open systems
    - Magnetohydrodynamic approach
    - Field line coordinates
    - Linear instabilities & quantum chaos
    - Drift wave turbulence and self-organization of zonal flows
  - Part 2: Current and open problems in multi-region relaxation theory

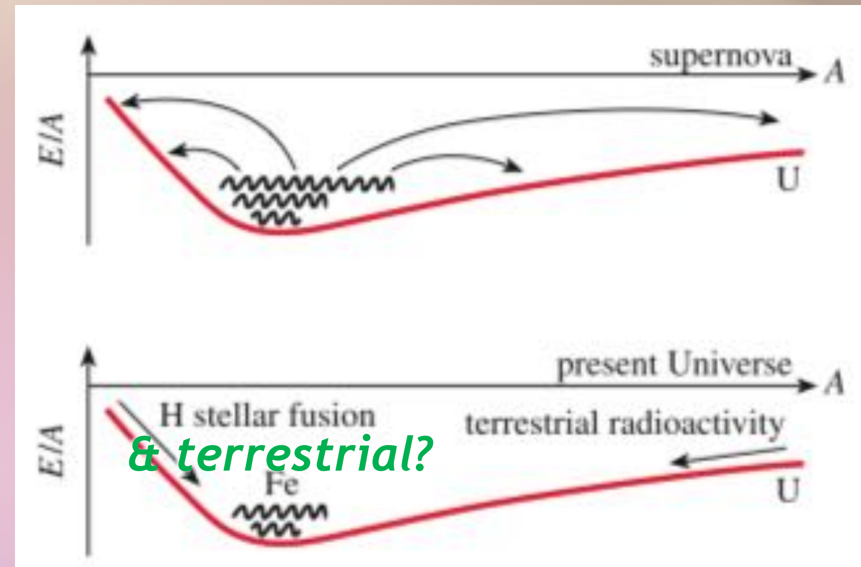
Plasma physics and fusion power

- Plasma is gas that is hot enough that some or all of the atoms are *ionized* i.e. lose their electrons and become electrically conducting

- Much of astrophysics concerns plasma and there are many industrial applications, but the "holy grail" is the generation of *fusion power* – abundant fuel, much lower radioactive waste than fission power

The Nuclear Valley

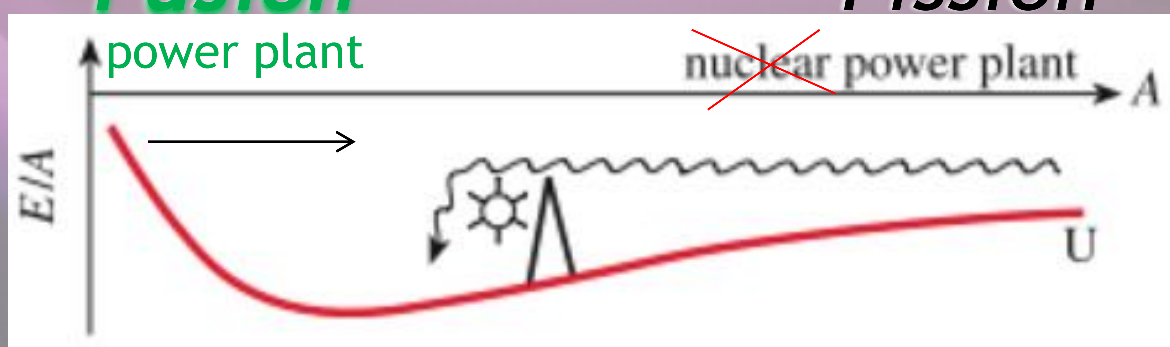




G. Marx: *Life in the nuclear valley* Phys. Educ. **36**, 375 (2001)

Fusion

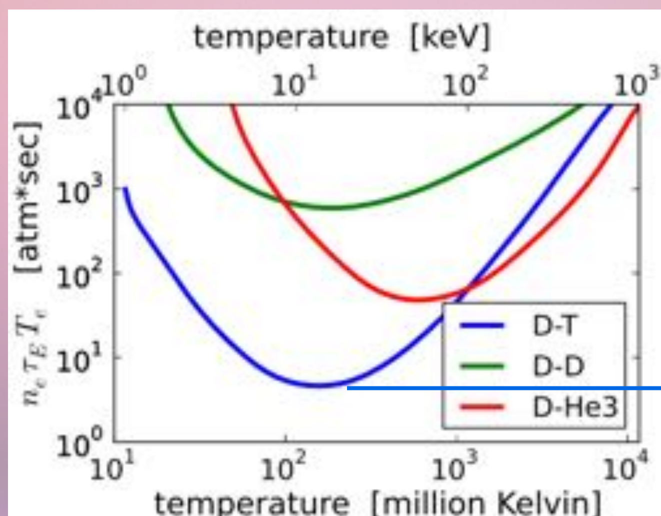
Fission



2 approaches to fusion power — inertial & *magnetic* confinement

- Lawson criterion: To get net energy production with D-T need triple product > 10

triple product

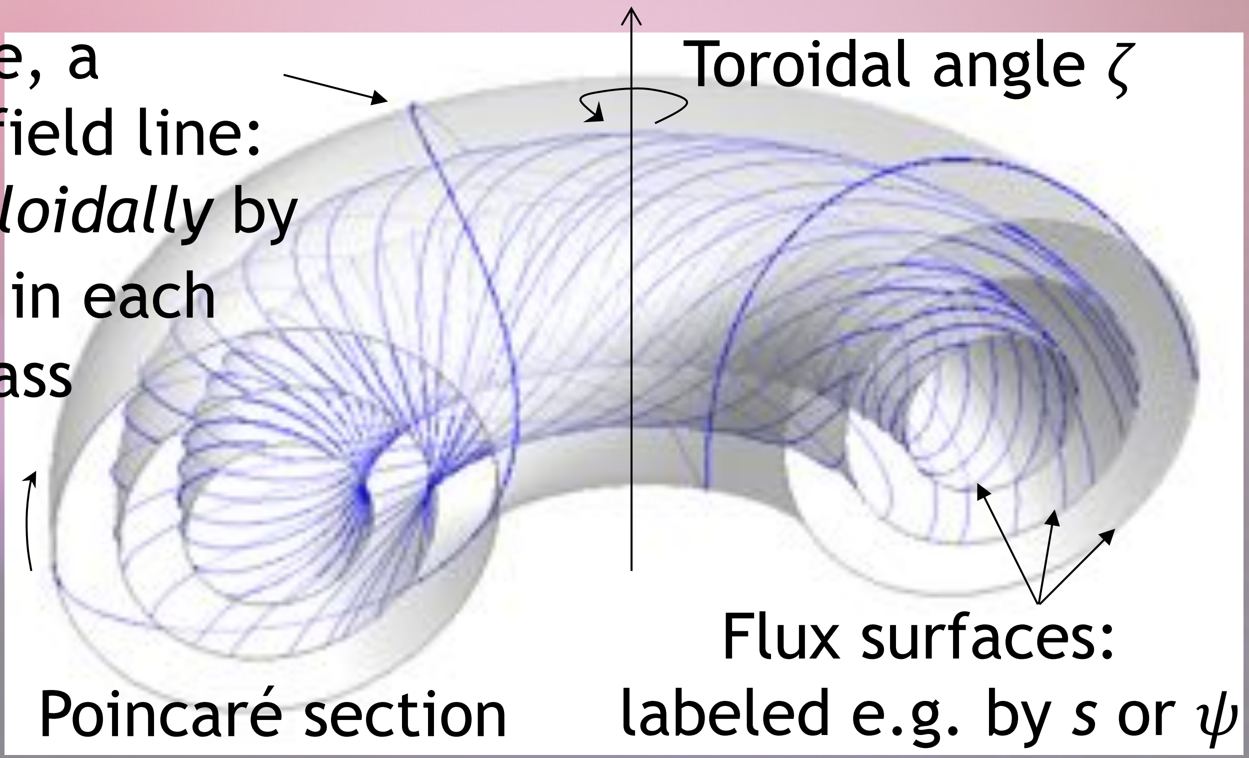


Temperature > 10^4 K, i.e. 10 keV

- Inertial confinement: short time, high pressure
- *Magnetic confinement*: long time, low (1 atm) pressure

Magnetic field geometry

- Toroidal magnetic confinement requires both *toroidal* and *poloidal* field components ⇨ *helically* twisting field lines
- Example shown: axisymmetric system (tokamak)



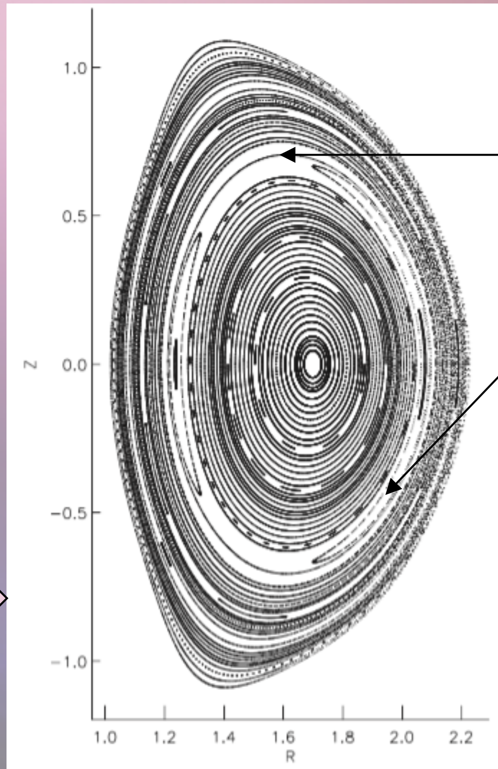
Tokamak plasmas are often said to be *doughnut shaped*

"Well, what <u>do</u> you say to a person who tells you he's working on a doughnut–shaped energy field?"

Magnetic field "dynamics"

- Motion on field lines is dynamical system $\dot{\mathbf{r}} \equiv d\mathbf{r}/d\zeta \propto \mathbf{B}$ where ζ is a toroidal angle
- System is Hamiltonian (Morrison, March 22)
- Analyze with flux-preserving *return map* to Poincaré section ζ = 0:

*Non*axisymmetric (3-D) fields are not generically *integrable*, i.e. some points do not lie exactly on flux surfaces (e.g. KAM* *invariant tori*). Instead follow *chaotic* orbits in island separatrices.
*Kolmogorov, Arnol'd & Moser



invariant torus: ι = irrational

island: $\iota = 1/2$

ι = (average rotation angle per return)/2π is *rotation number*

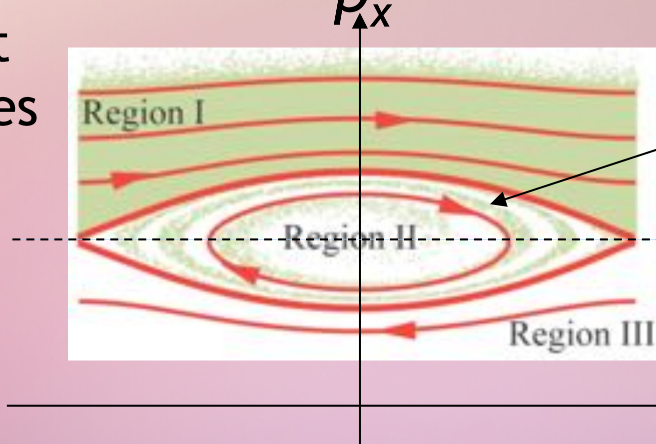
Particle dynamics

- Electrons in e.s.* wave $H = \frac{p_x^2}{2m} - e\varphi_0 \cos(kx - \omega t)$

Like physical pendulum

Region I: Fast passing particles

Resonant velocity: $\frac{p_x}{m} = \frac{\omega}{k}$



Region II: Trapped particles in phase-space island

Region III: Slow passing particles

**electrostatic*

- Ions in a tokamak

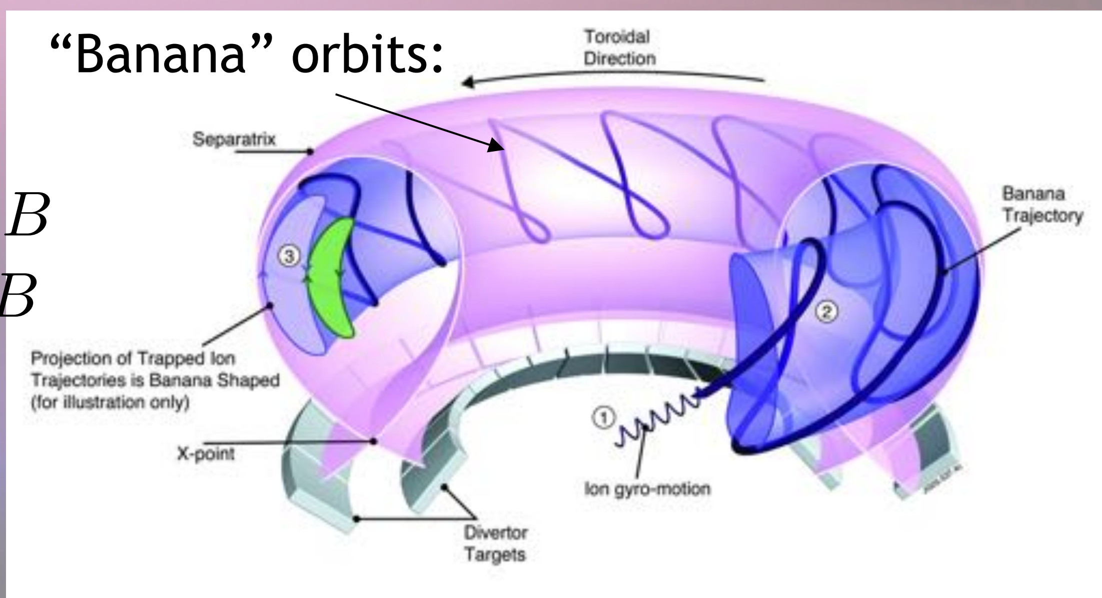
Obey *drift equations*:

Adiab. invariant $\mu = mv_\perp^2/2B$

Energy invariant $mv_\parallel^2/2 + \mu B$

Cross-field ∇B drift

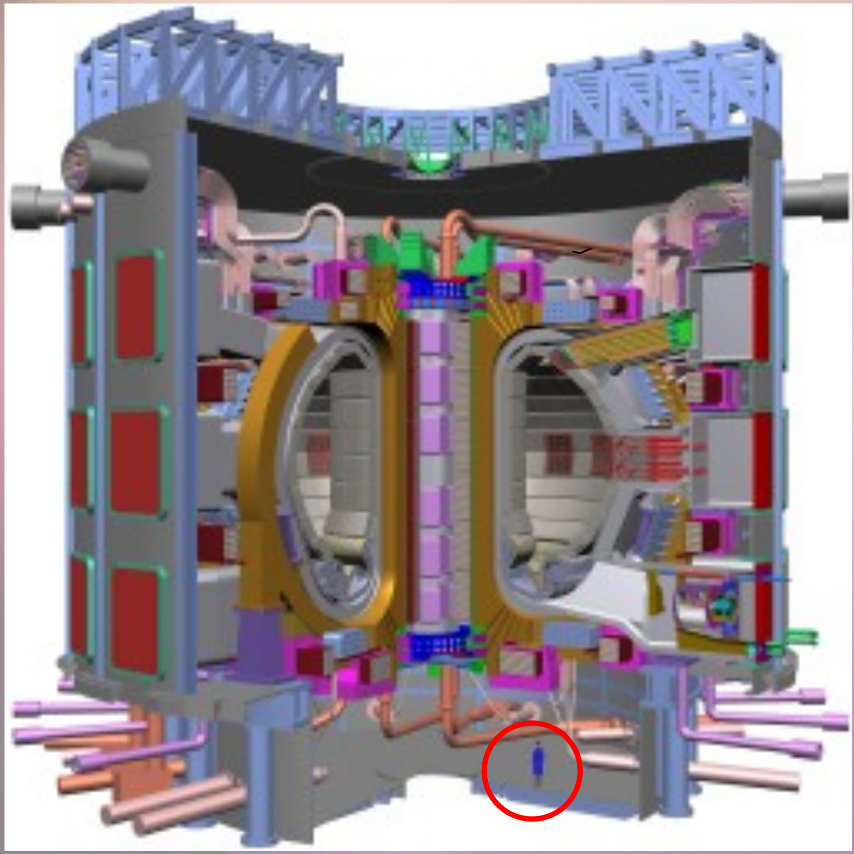
$$\mathbf{v}_{\nabla B} = \frac{\mu}{q}\frac{\mathbf{B}\times\boldsymbol{\nabla}B}{B^2}$$



The need for helical twist

- The ∇B drift is also the reason why we need both toroidal and poloidal field:
- Having both makes field lines helical
- As conductivity is high along field lines this makes surfaces equipotentials and stops plasma drifting to the wall
- *Tokamaks* use a *high toroidal current* to generate the poloidal field, but this can cause dangerous disruptions
- *Stellarators* use 3-D geometric effects instead, thus much more robust against disruption

Largest magnetic confinement fusion experiment — ITER



http://www.iter.org/

International burning plasma experiment **ITER** under construction in Cadarache, France

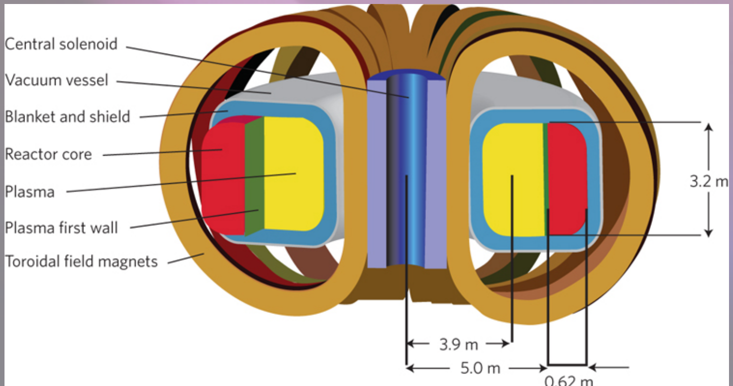
Partners: Europe, <u>Japan</u>, China, India, Russia, South Korea, USA

Some other alternative approaches (some privately funded!)

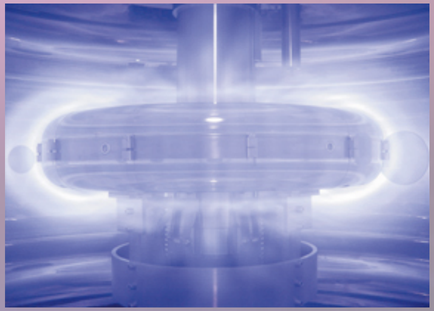
- Spherical tokamaks (MAST,NSTX) Compact, high-β, where $\beta \equiv \left\langle \frac{2\mu_0 p}{B^2} \right\rangle$

- Reversed field pinches: low toroidal field, self-organized helical states (RFX, Padua)

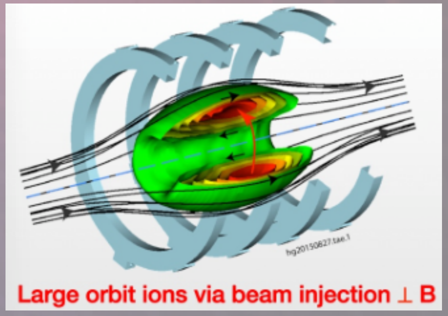
- Fusion-fission hybrids, technically attractive, but not politically?



- Aneutronic ${}^{1}p^{11}B$ rather than DT?



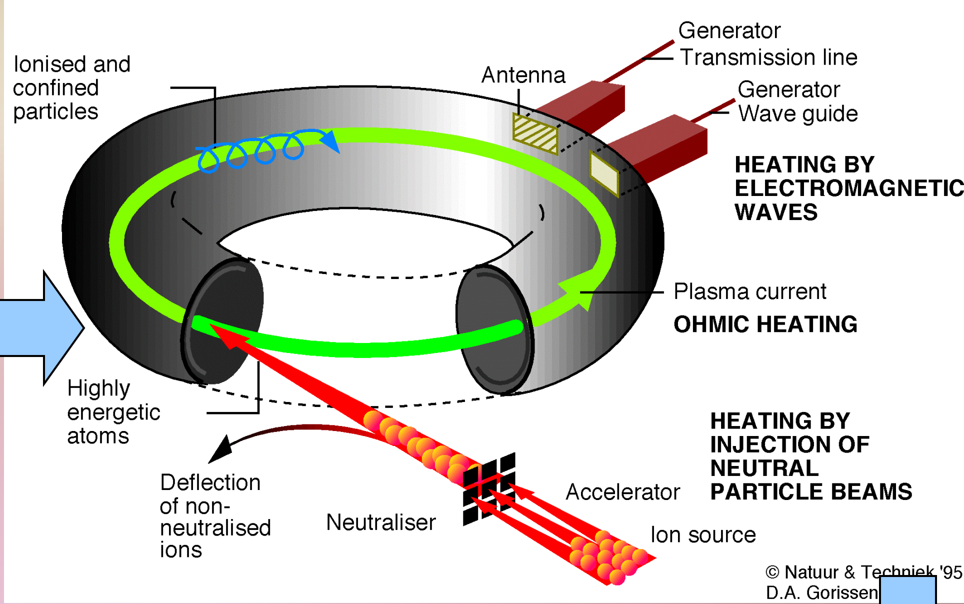
RT-1 levitated dipole, Yoshida Lab, Kashiwanoha



TRI ALPHA ENERGY
THE POWER OF INGENUITY

Field-Reversed Config., Irvine, USA

Open systems:

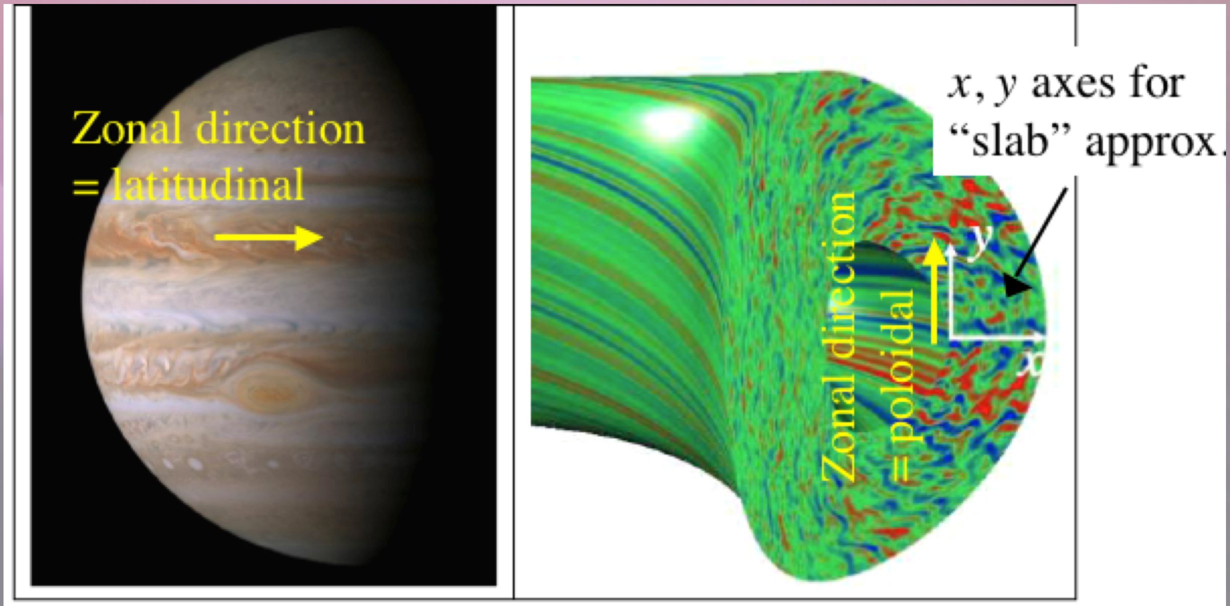


- "Heat" in:
  - Ohmic, rf, neutral & beam heating
  - fusion reaction products: 3.5 MeV ^{4}He ions (α particles)
- Matter in:
  - neutral beams
  - pellet injection
  - recycling from walls via recombined hydrogen atoms + sputtered impurities

- "Heat" out:
  - thermal (anomalous) diffusion from ~10^{8}°K @ centre of plasma to ~10^{3}°K @ walls
  - 14 MeV neutrons
  - X-radiation
- Matter out:
  - diffusion & advection of plasma to walls

Complex systems

- Strongly driven systems exhibit turbulence, but also self-organization
- Cross-disciplinary — theory pushes boundaries of statistical mechanics, nonlinear dynamics, fluid dynamics, …



Some approaches to complex systems analysis

- Reduce nonequilibrium *N*-body problem to *continuum* models (fluid, Vlasov etc)
- Find *equilibria & periodic orbits* in simple cases and use nonlinear *continuation* to follow them to more complex cases
- Linearize about these and determine stable and unstable normal modes
- Locate stability thresholds/bifurcation points
- Find reduced-dimensionality models on slow manifolds
- Use simulations to find phenomena missed by above

Fluid models: Ideal Magnetohydrodynamics (MHD)

- Mass and entropy in *each fluid element* are conserved by mass ρ & pressure p advection equations

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\rho = -\rho\boldsymbol{\nabla}\cdot\mathbf{v}$$

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})p = -\gamma p\boldsymbol{\nabla}\cdot\mathbf{v}$$

- Magnetic flux threading *each microscopic loop* advected by the flow is also conserved ("frozen in") by magnetic field **B** advection equation

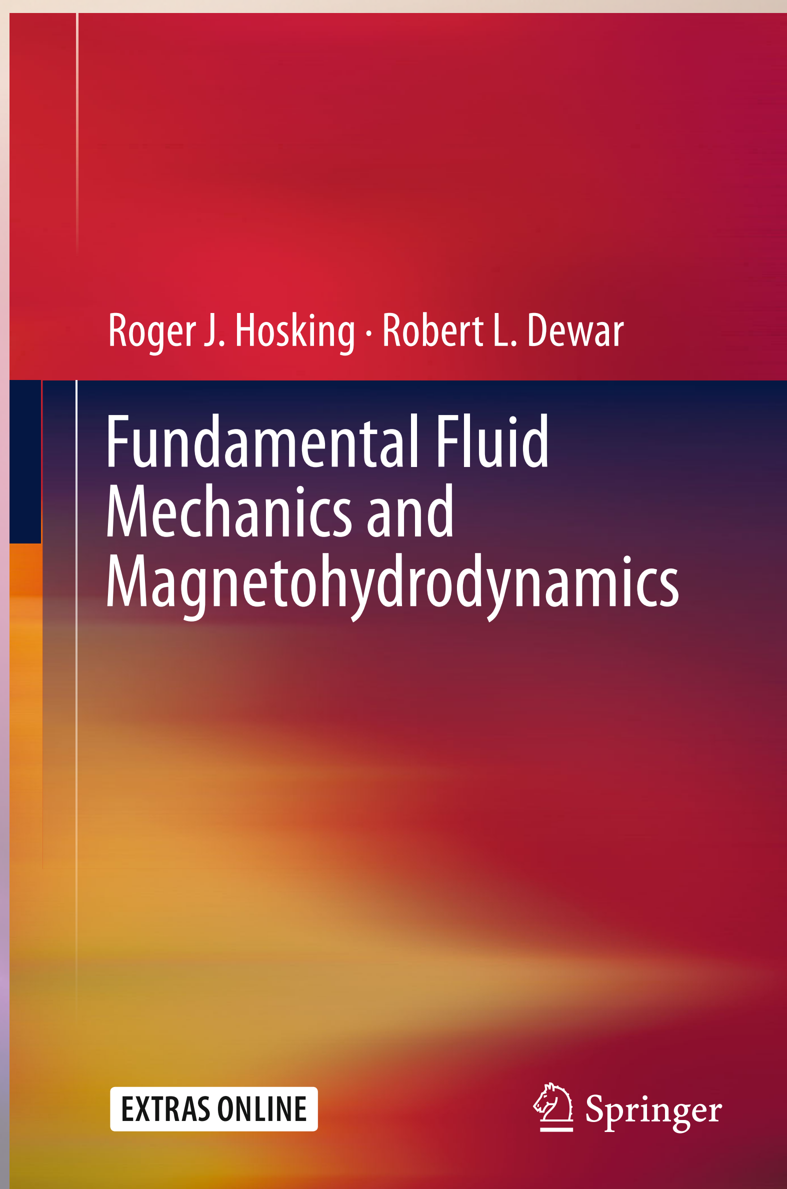
$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} = -\mathbf{B}\cdot(\mathsf{I}\boldsymbol{\nabla}\cdot\mathbf{v} - \boldsymbol{\nabla}\mathbf{v})$$

- Fluid equation of motion

$$\rho(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}\cdot\left[\left(p + \frac{B^2}{2\mu_0}\right)\mathsf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}\right]$$

NB Divergence of stress tensor includes the $\mathbf{j}\times\mathbf{B}$ force

New book(© 2016)



MHD waves & instabilities

- Local dispersion relations for large $|\mathbf{k}|$, small $k_{\parallel}/k_{\perp}$ (relative to direction of **B**):

Alfvén wave: $$\rho\,\omega_{\mathrm{A}}^2 = (\mathbf{k}\cdot\mathbf{B})^2$$

Slow magneto-sonic wave: $$\rho\,\omega_{\mathrm{S}}^2 = \frac{\gamma p\,(\mathbf{k}\cdot\mathbf{B})^2}{\mu_0^{-1}B^2 + \gamma p}$$

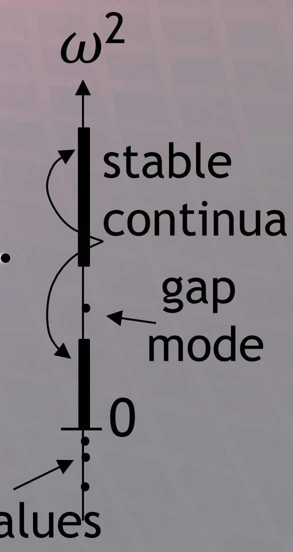
(Alfvén and slow waves:) $\omega \rightarrow 0$ as $k_{\parallel} \rightarrow 0$; $\mathbf{v}_{\mathrm{g}} \equiv \partial\omega_{\mathbf{k}}/\partial\mathbf{k} \,||\, \mathbf{B}$

characteristics propagate information *parallel* to field

Fast magneto-sonic wave: $$\rho\,\omega_{\mathrm{F}}^2 = (\mu_0^{-1}B^2 + \gamma p)k^2$$

NB $\omega_{\mathrm{F}}{}^2 \gg 0$: low-$\omega$ modes cannot involve fast mode.

- Normal modes of flow-free equilibria are found from eigenvalue problem $\rho\omega^2\boldsymbol{\xi} = -\mathbf{F}\cdot\boldsymbol{\xi}$, where the linearized force operator **F** is *Hermitian*: eigenvalues ω^2 are *real*, so instability (imaginary ω) only if $\omega^2 < 0$.
- Thus *stability threshold* is $\omega = 0$. (*NB* ξ denotes fluid element displacement. Take $\mathbf{k}\cdot\boldsymbol{\xi} = 0$ to exclude fast mode.)



Helically fluted columns/waves

- A *flute mode* in toroidally confined plasma has constant phase $(\mathbf{k} \cdot \mathbf{B} = 0)$ along the helical magnetic field lines — nothing to do with the the musical instrument!

Great Colonnade at Greco-Roman ruins in Apamea, Syria, showing *helically fluted columns*

Ideal-MHD short-wavelength flute-type instabilities

- All 3 branches of local dispersion relation have $\omega^2 \geq 0$, so are stable; but Alfvén and slow m.s. modes get to instability *threshold* $\omega^2 = 0$ when $\mathbf{B} \cdot \mathbf{k} = 0$, implying $\mathrm{B}{\cdot}\boldsymbol{\nabla}\boldsymbol{\xi} \approx 0$ for unstable MHD modes *beyond standard WKB*.
- Suggests *anisotropic* WKB ansatz

$$\boldsymbol{\xi} = \hat{\boldsymbol{\xi}} \exp[iS(\mathbf{r})/\epsilon - i\omega t], \;\; \mathbf{B}{\cdot}\boldsymbol{\nabla} S \equiv 0, \;\; \hat{\boldsymbol{\xi}} \, \text{slowly varying}$$

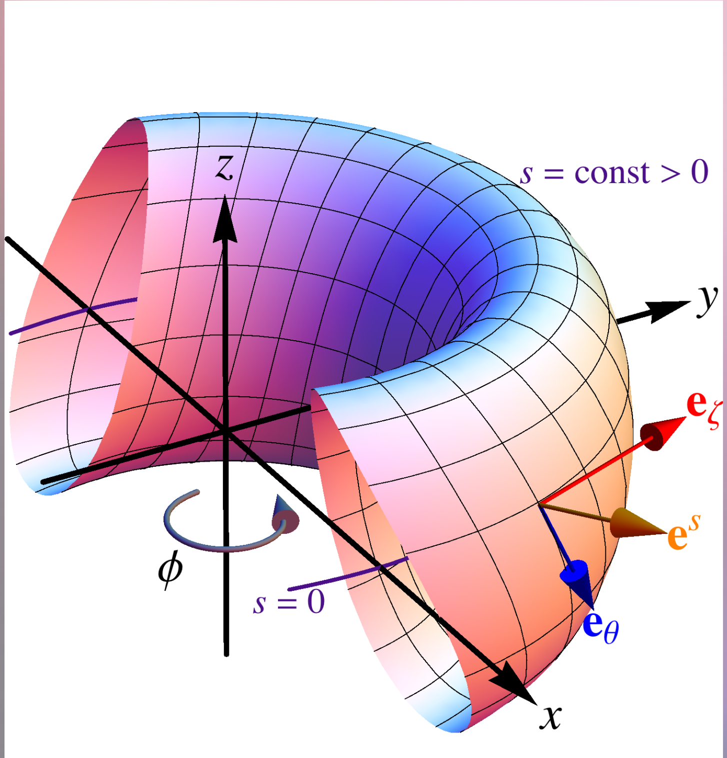
where ϵ is inserted for formal asymptotic analysis. When ray dynamics is integrable, global spectrum found from EBK (e.g. semiclassical, Bohr-Sommerfeld) quantization (see later).

to implement, need special *curvilinear coords*:

Ref: R.L. Dewar & A.H. Glasser, Phys. Fluids **26**, 3038 (1983)

Straight-Field-line coordinates

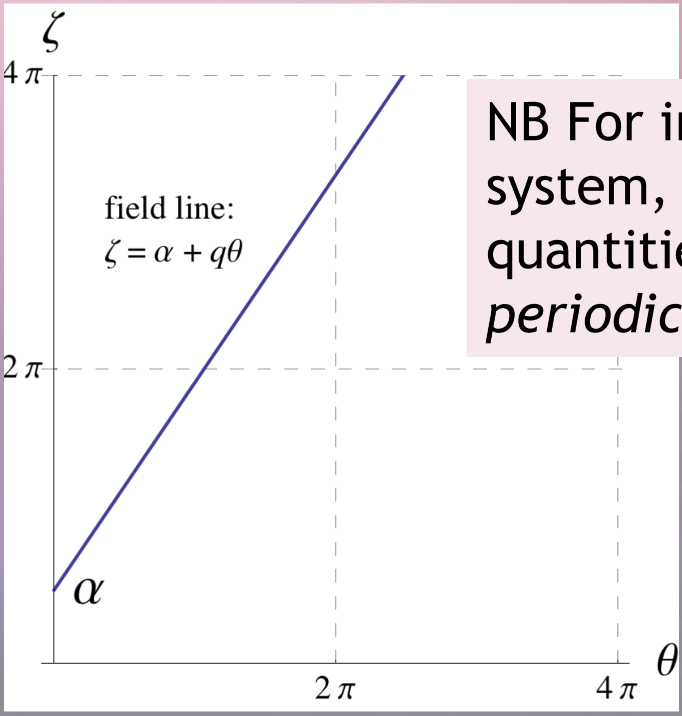
- On each toroidal magnetic surface use *generalized* ($\neq \phi$) toroidal angle ζ and/or generalized poloidal angle θ such that field lines are straight:



Curvilinear coordinates s, θ, ζ

On θ, ζ *covering space* on a magnetic surface* $s = \text{const}$, field lines are *straight lines* $\alpha \equiv \zeta - q(s)\theta = \text{const.}$

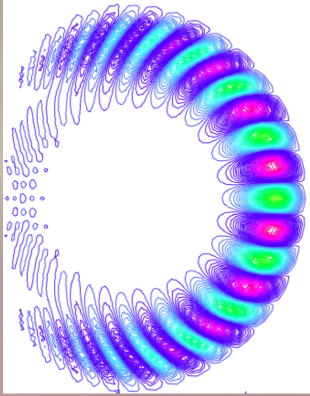
$q \equiv 1/\iota$



NB For irrational *q* in a 3-D system, equilibrium quantities change *quasi-periodically* along field line

*Open Question: What is “best” approximation to a broken surface in a 3-D system

Ballooning/interchange instabilities



- Short-perpendicular-wavelength instabilities can be found using the anisotropic WKB ansatz $\varphi = \widehat{\varphi}\exp(iS/\epsilon - i\omega t)$, where $\boldsymbol{\xi}_\perp = \frac{\mathbf{B}\times\boldsymbol{\nabla}\varphi}{B^2}$, giving the *ballooning equation*:

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{k}^2}{\mu_0 B^2}\mathbf{B}\cdot\boldsymbol{\nabla}\varphi\right) + \frac{2\mathbf{k}\cdot\boldsymbol{\nabla}p\times\mathbf{B}\,\boldsymbol{\kappa}\times\mathbf{B}\cdot\mathbf{k}}{B^4}\varphi + \frac{\rho\mathbf{k}^2}{B^2}\omega_k^2\varphi = 0$$

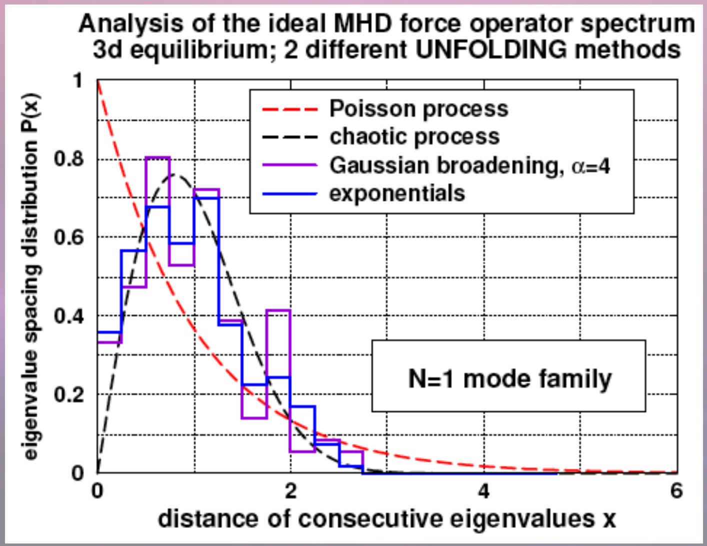
$\boldsymbol{\kappa} \equiv \mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b}$ is the *field-line curvature vector*, and $\mathbf{b}$ is the parallel unit vector $\mathbf{b} \equiv \mathbf{B}/B$.

This is an *ordinary* differential equation to be integrated along each infinite field line $\alpha = \text{const}$. Requiring that φ decay exponentially at $\pm\infty$ gives an eigenvalue problem determining the *local dispersion relation* $\omega = \omega_k(\alpha, \theta_k)$, where $\theta_k \equiv kq/k_\alpha$. Decay can be due to magnetic shear, or, in 3-D geometry, *Anderson localization* due to quasiperiodicity.

Classical Quantum Chaos!

- *Quantum* chaos refers to random-looking eigenvalue spectrum occurring when *ray equations*, $\dot{\mathbf{x}} = \partial\omega_{\mathbf{k}}/\partial\mathbf{k}$, $\dot{\mathbf{k}} = -\partial\omega_{\mathbf{k}}/\partial\mathbf{x}$, exhibit *classical* chaos in **x**,**k** phase space when rays are *bounded but not integrable*.
- Eigenval. spacings obey Wigner distribution:

Analysis of a W7-X interchange instability spectrum shows q-chaos signature: Wigner distribution



*OPEN QUESTION:
Is MHD continuum (*stable* modes) analogous to quantum continuum (*unbound* states)? Can quantum chaotic *scattering* theory be used in MHD?

Hasegawa-Mima equation

The Hasegawa-Mima equation provides the simplest model for propagation of *drift waves* in magnetically confined plasmas. It is, as originally written, isomorphic to the geostrophic vorticity (Charney–Obukhov) equation, with *Rossby waves* the analogues of drift waves.

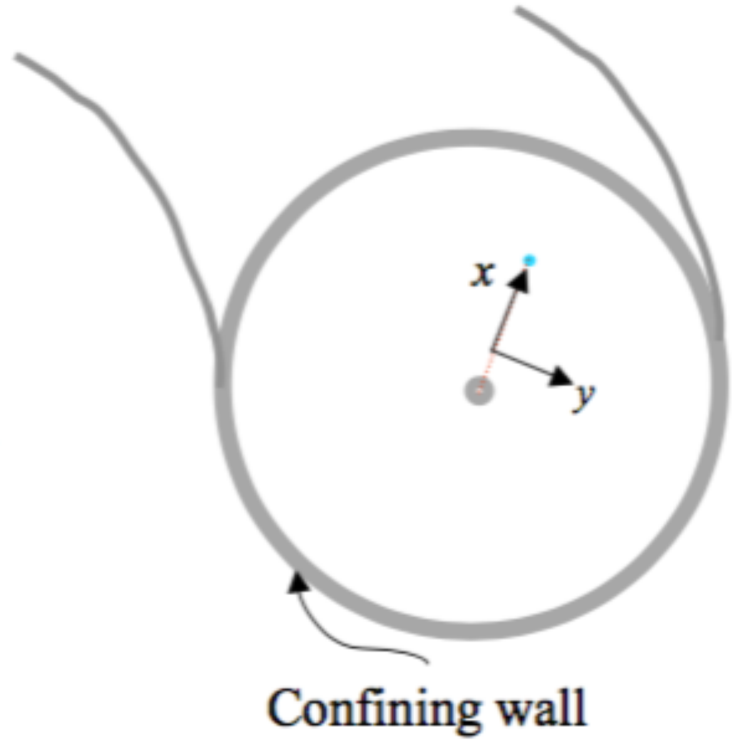
$$\left(\partial_t + \mathbf{v}_E \cdot \nabla + \mathbf{v}_* \cdot \nabla\right)\psi - \left(\partial_t + \mathbf{v}_E \cdot \nabla\right)\rho_s^2 \nabla^2 \psi = 0$$

$\mathbf{v}_E = \hat{\mathbf{z}} \times \nabla \psi$: E×B drift

$\mathbf{v}_* \equiv -\dfrac{T_e \hat{\mathbf{z}} \times \nabla n_0}{e B_0 n_0}$: Diamagnetic drift due to background inhomogeneity

$\rho_s \equiv \omega_{\mathrm{ci}}^{-1} (T_e / m_i)^{1/2}$: "Effective" Ion Larmor radius

$\omega_{\mathrm{ci}} \equiv e B_0 / m_i$: Cyclotron frequency



Modified Hasegawa-Mima equation

$$(\partial_t + \mathbf{v}_E \cdot \nabla + \mathbf{v}_* \cdot \nabla)\tilde{\psi} - (\partial_t + \mathbf{v}_E \cdot \nabla)\rho_s^2 \nabla^2 \psi = 0$$

$\tilde{\psi} = \psi - \overline{\psi}$ zonal (*y*-averaged) part is subtracted from the potential/stream function to give fluctuation

$$\overline{\psi} = \frac{1}{L_y} \int_0^{L_y} \psi \, d_y$$

The modification is particularly important in studying drift wave-zonal flow interplay.

[W. Dorland et al., Bull. Am. Phys. Soc. 35, 2005 (1990),
W. Dorland and G. Hammett, Phys. Fluids B 5, 812 (1993)]

Resistive drift waves: Hasegawa-Wakatani equations

$$\{a, b\} \equiv (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x)$$

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

$$\zeta \equiv \nabla^2 \varphi$$

$$\frac{\partial}{\partial t}\zeta + \{\varphi, \zeta\} = \alpha(\varphi - n) - D\nabla^4\zeta,$$

$$\frac{\partial}{\partial t}n + \{\varphi, n\} = \alpha(\varphi - n) - \kappa\frac{\partial \varphi}{\partial y} - D\nabla^4 n.$$

Hyperviscosity/diffusion to suppress high k

Drift-wave driving term — density gradient

$$\kappa \equiv (\partial/\partial x)\ln n_0.$$

adiabaticity operator $\alpha \equiv -T_e/(\eta n_0 \omega_{ci} e^2)\partial^2/\partial z^2$

- arises from Ohm's law:

$$j_z = -env_{e,z} = -\frac{1}{\eta}\frac{\partial}{\partial z}\left(\varphi - \frac{T_e}{e}\ln n\right)$$

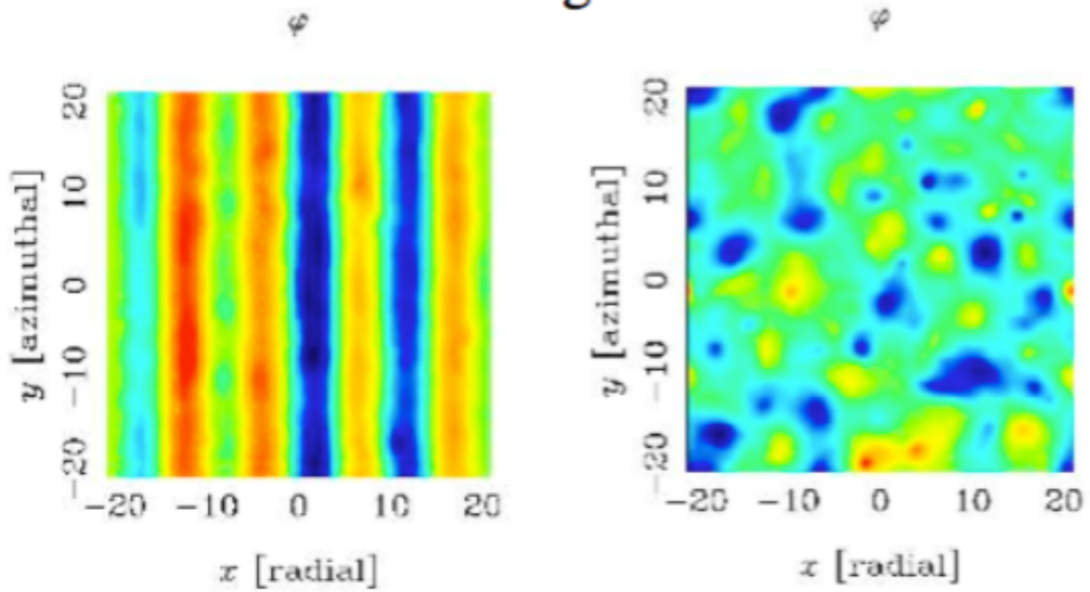
- becomes parameter by assuming typical k_z

- retrieve Hasegawa–Mima eqn. in limit $\alpha \rightarrow \infty$
- get normal fluid eqn. in limit $\alpha \rightarrow 0$

Inverse cascade of Modified HW turbulence produces zonal flows

- Dorland-Hammett modification leads to preferential growth of “zonal” modulations (which can suppress turbulent transport)
  Numata, Ball & Dewar, Phys. Plasmas **14**, 102312 (2007)

Modification enhances generation of zonal flows



Conclusion of Part 1

- Have tried to indicate broad scope of plasma theory for fusion physics
- Three-dimensional geometries pose many mathematical problems
- Self-organization of turbulent plasmas can give rise to transport barriers