Lecture Note *

Self-organized structures over scale hierarchy: non-canonical Hamiltonian mechanics, phase space foliation, and "vortex"

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December 4, 2012

Abstract

Self-organization of structures is one of the most challenging problems of modern physics. Staring from standard classical mechanics, we develop an advanced framework of mechanics that has "non-canonical" foliated phase space. Vortical structures in the Universe are explained as creations on leaves of macroscopic scale hierarchy. This course provides students with advanced level of theoretical understanding and mathematical methods applicable in a wide area of nonlinear sciences.

*This lecture was given at Ivane Javakhishvili Tbilisi State University on 22nd, 23rd, 26th, 27th, 28th, and 29th of November, 2012.

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1 Scale-hierarchy and Self-organization

Self-organization of a structure is, at its surface, an antithesis of the entropy *ansatz*. However, disorder can still develop at microscopic scale while a structure emerges on some macroscopic scale; it seems more common in various nonlinear systems that order and disorder are simultaneous, and such co-existence may be possible if the self-organization and the entropy principle work on different scales. Therefore we have to write a theory of self-organization as a discourse on *scale hierarchy*.

Indeed scale hierarchy is a popular keyword in various arguments on "structures"; a biological body is a typical example in which an evident hierarchical structure is programmed to establish, enabling effective consumption of energy and materials as well as emission of entropy and wasts. But the theory of a physical macro-system —a collective system of "simple" elements, like a gravitational system or a plasma— hinges on a different framework; a scale hierarchy is not "programed" to emerge, or structures are not subject to some functions; yet one can observe a more fundamental and elementary process of *creation* in nonlinear dynamics.

For example, magnetospheres are self-organized structures found commonly in the Universe. In the vicinity of a dipole magnetic field rooted in the central object, a plasma clump with rather steep density gradient is created (Fig. 1). So-called *inward diffusion* (or up-hill diffusion) drives charged particles toward the inner higher-density region, which is seemingly opposite to the natural direction of diffusion (normally, diffusion is a process of flattening distributions of physical quantities). [1] Creation of such a macroscopic structure can be explained only by delineating a fundamental difference of macroscopic hierarchy and the conventional microscopic (or scale irrelevant) narrative of physics.

To set a stage for the discussion of scale hierarchy, let us review the usual Boltzmann distribution and show how the self-organization of a magnetospheric plasma clump is "strange" in the view of microscopic framework. The energy of a charged particle is a sum of the kinetic energy and the potential energy:

$$H = \frac{m}{2}v^2 + q\phi, \tag{1}$$

[1]The process is driven by some spontaneous fluctuations (symmetry breaking) that violates the constancy the the angular momentum; in a strong enough, symmetric magnetic field, the canonical angular momentum P_θ is dominated by the charge q multiple of the flux function ψ (Gauss' potential of the magnetic field), thus the conservation of $P_\theta \approx q\psi$ constrains the charged particle on a magnetic surface (level-set of ψ). Perturbed by a random-phase fluctuations, particles can diffuse across magnetic surfaces.

where $\boldsymbol{v} := (\boldsymbol{P}-q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential, m is the mass, and q is the charge. The standard Boltzmann distribution function is derived when we assume that the Lebesgue measure d^3vd^3x is an invariant measure and the Hamiltonian H is the determinant of the ensemble; maximizing the entropy

$$S = -\int f \log f d^3vd^3x \tag{2}$$

under the constraints on the total energy $E = \int Hfd^3vd^3x$ and the total particle number $N = \int fd^3vd^3x$, we obtain

$$f(\boldsymbol{x}, \boldsymbol{v}) = Z^{-1}e^{-\beta H}, \tag{3}$$

where Z is the normalization factor ($\log Z - 1$ is the Lagrange multiplier on N) and β is the inverse temperature (the Lagrange multiplier on E). The corresponding configuration-space density is

$$\rho(\boldsymbol{x}) = \int fd^3v \propto e^{-\beta q\phi}, \tag{4}$$

which becomes homogeneous if the charge neutrality condition applies ($\phi \equiv 0$).

The puzzle of creation of magnetospheric plasma clump is not solved by inquiring into the Hamiltonian; since magnetic field does not cause any change in the energy of particles, there is no way to revise the energy in the calculation of the equilibrium state. Instead, it is solved by finding an appropriate "phase space" (or an ensemble) on which the Boltzmann distribution is achieved; the identification of an appropriate macroscopic phase-space is nothing but the formulation of what we call "scale hierarchy".

We describe the scale hierarchy by a phase-space foliation, and explain the self-organization (creation of heterogeneity) by a distortion of the metric (invariant measure) on leaves. In the foregoing example of magnetospheric plasma confinement, the phase-space of magnetized particles is constrained (foliated) by adiabatic invariants, and the metric on each leaf is distorted with respect to the laboratory-fame flat space; the particles distribute homogeneously on the leaf can have peaked profile when the inhomogeneous Jacobian weight multiplies.

The aim of this lecture is to show how a macroscopic scale hierarchy is foliated, and how interesting "structures" are created on the leaves. Foliation of phase space is due to "noncanonicality" of the geometry that governs the dynamics. Aforementioned example of "self-organized confinement" will

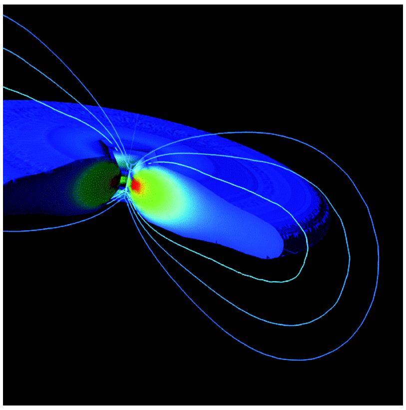

Figure 1: Jovian magnetosphere (theoretical model by J. Shiraishi, S. Ohsaki, and Z. Yoshida, Phys. Plasmas **12** (2005), 092901).

be discussed as an example finite-dimensional systems. We will also study infinite-dimensional systems by referring examples of fluid and plasma dynamics.

2 The Framework of Hamiltonian Mechanics

In this section, we review the general framework of Hamiltonian mechanics, and see how symplectic geometry dictates the dynamics. Some mathematical backgrounds are given in Appendix A.

2.1 Classical mechanics (Hamiltonian formalism)

We start by reviewing the standard symplectic geometry and Hamiltonian mechanics. We denote by $\boldsymbol{z} = (q^1, \cdots, q^m, p^1, \cdots, p^m)$ the *state vector*, a "point" in an affine space $X = \mathbb{R}^{2m}$ (to be called *phase space*). A canonical Hamiltonian system is endowed with a *Hamiltonian* $H(\boldsymbol{z})$ (a real function on the phase space X) and an $2m \times 2m$ antisymmetric regular matrix

$$J_c := \begin{pmatrix} 0_m & I_m \\ -I_m & 0_0 \end{pmatrix},$$

where I_m and 0_m are the m-dimensional identity and nullity. [2] We call J_c a *canonical Poisson operator* (matrix). The equation of motion (Hamilton's equation) is written as

$$\frac{d}{dt}\boldsymbol{z} = J_c \partial_{\boldsymbol{z}} H(\boldsymbol{z}). \tag{5}$$

We call a scalar function on the phase space X (i.e. a functional $f : X \to \mathbb{R}$) an *observable.* The totality of observables is denoted by $\mathrm{Fun}(X)$. [3] Defining a *Poisson bracket* by

$$\{a, b\} := (\partial_{\boldsymbol{z}} a, J_c \partial_{\boldsymbol{z}} a) = (\partial_{z^i} a)\mathcal{J}_{ij}(\partial_{z^j} b),$$

we may evaluate the rate of change of an observable $f(\boldsymbol{z})$ by

$$\frac{d}{dt} f = \{f, H\}.$$

The Poisson bracket is a bilinear derivative[4] map to a scalar function ($\{\ ,\ \} : \mathrm{Fun}(X) \times \mathrm{Fun}(X) \to \mathrm{Fun}(X)$), [5] defining a Lie algebra on $\mathrm{Fun}(X)$.

[2] In what follows, we may write just I or 0 without specifying the dimension (especially when we consider an infinite-dimensional space).

[3] $\mathrm{Fun}(X)$ is a commutative ring endowed with the conventional vector calculus and the associative products of scalar functions.

[4] A Lie bracket satisfying the *derivative relation* $\{a, bc\} = \{a, b\}c + \{a, c\}b$ is called a Poisson bracket.

[5] Notice that, when we write the Poisson bracket as $\{a, b\} = (\partial_{\boldsymbol{z}} a, J_c \partial_{\boldsymbol{z}} b)$, the left-hand side is a new scalar function $\in \mathrm{Fun}(X)$ (not a real number evaluated by the right-hand-side functional at a single point $\boldsymbol{z}$, but a function defined over X).

The *Hamiltonian flow* is a *vector* in the phase space: Regarding ∂_{z^i} as the basis of the tangent space,

$$\mathrm{ad}(H) := -\{H,\ \} = J_c^{ij}(\partial_{z^j} H)\partial_{z^i}.$$

Note 1 (Differential operators) *We can also regard* $\underset{\sim}{a} := \mathrm{ad}(a)$ *as a* differential operator *(*$\mathrm{Fun}(X) \to \mathrm{Fun}(X)$*). Let us denote by* $\mathcal{D}(X)$ *the linear space of such differential operators. We can define a Lie-ring structure on* $\mathcal{D}(X)$ *by the commutation of non-commutative products* $\underset{\sim}{a}\underset{\sim}{b}$ *and* $\underset{\sim}{b}\underset{\sim}{a}$*:*

$$[\underset{\sim}{a}, \underset{\sim}{b}] := \underset{\sim}{a}\underset{\sim}{b} - \underset{\sim}{b}\underset{\sim}{a} = \mathrm{ad}(a)\mathrm{ad}(b) - \mathrm{ad}(b)\mathrm{ad}(a).$$

By Jacobi's identity

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0, \tag{6}$$

we find

$$[\underset{\sim}{a}, \underset{\sim}{b}] = \mathrm{ad}(\{a, b\}).$$

The $\mathrm{ad}(H)$ *is the* adjoint representation *of the Lie algebra defined by the Poisson bracket (so called* Poisson algebra*). The Lie bracket* $[\underset{\sim}{a}, \underset{\sim}{b}]$ *defines a Lie ring* $\mathcal{A}$ *(*$\mathcal{D}(X)$ *is the enveloping algebra of* $\mathcal{A}$*, i.e.* $U(\mathcal{A}) = \mathcal{D}(X)$*).*

2.2 Classical mechanics on T^*M

We generalize the space $\mathbb{R}^{2m}$ to a general cotangent bundle T^*M of a smooth manifold M of dimension m. The canonical Poisson operator J_c is related to the *symplectic 2-form* that determines the geometric structure of canonical Hamiltonian system derived by an *action principle*. A symplectic 2-form can be represented by local coordinates as

$$\omega = dq^1 \wedge dp^1 + dq^2 \wedge dp^2 + \cdots + dq^n \wedge dp^n, \tag{7}$$

which is the "vorticity" (or a "field tensor") of a canonical 1-form

$$\theta = p^1 dq^1 + p^2 dq^2 + \cdots + p^n dq^n, \tag{8}$$

i.e. $\omega = d\theta$. Denoting

$$z^j = q^j, \quad z^{m+j} = p^j \quad (j = 1, \cdots, m),$$

we may write

$$\omega = \frac{1}{2} J^c_{k\ell} dz^k \wedge dz^\ell, \tag{9}$$

where $J^c = J_c^{-1} = -J_c$.

Hamilton's equation of motion is produced by an *action principle.* Given a Hamiltonian $H(\boldsymbol{z}, t)$, the action is an integral along a curve $\boldsymbol{z}(t)$ connecting a fixed start point $a = (\boldsymbol{z}_0, t_0)$ and end point $b = (\boldsymbol{z}_1, t_1)$:

$$S = \int_{t_0}^{t_1} \left(p^j \frac{dq^j}{dt} - H \right) dt. \tag{10}$$

The variation of S with respect to $\boldsymbol{z}(t) \to \boldsymbol{z}(t) + \epsilon \tilde{\boldsymbol{z}}(t)$ is

$$\delta_{\tilde{\boldsymbol{z}}(t)} S = \epsilon \int_a^b \left(J^c_{k\ell} \frac{dz^\ell}{dt} - \frac{\partial H}{\partial z^k} \right) \tilde{z}^k dt + O(\epsilon^2),$$

The Euler-Lagrange equation is

$$J^c_{k\ell} \frac{dz^\ell}{dt} = \frac{\partial H}{\partial z^k}. \quad (k = 1, \cdots, 2n) \tag{11}$$

By multiplying $J_c^{-1} = -J_c$, we obtain

$$\frac{dz^\ell}{dt} = J_c^{k\ell} \frac{\partial H}{\partial z^k} \quad (k = 1, \cdots, 2n), \tag{12}$$

which is the canonical form of (5).

The geometry determined by the 2-form ω is, thus, invariant on the group

$$\mathrm{Sp}(m) = \{A \in \mathrm{GL}(2m, \mathbb{R}); A^{-1} J_c A = J_c\},$$

i.e. the symplectic geometry is endowed with a "G-structure" determined by the symplectic group $\mathrm{Sp}(m)$.

2.3 Generalized action principle

While we find that a symplectic geometry is naturally implemented on T^*M by canonical Hamiltonian mechanics (i.e. by a symplectic 2-form), there is an asymmetry between q^j (position on M) and p^j (coordinate of the cotangent space). To formulate a homogenized general action principle, we consider a general n-dimensional smooth manifold X (n may not be an even number), and consider a general non-degenerate closed 2-form such as

$$\omega = d\theta = d \left[\sum_{j=1}^{n} \theta_j(\boldsymbol{z}) dz^j \right].$$

Defining the "anti-symmetric field tensor"

$$A_{k\ell} := \frac{\partial \theta_\ell}{\partial z^k} - \frac{\partial \theta_k}{\partial z^\ell} \quad (1 \le k, \ell \le n), \tag{13}$$

we may write

$$\omega = \frac{1}{2} A_{k\ell} dz^k \wedge dz^\ell, \tag{14}$$

generalizing (9); when $n = 2m$, and $\theta_j = p^j$ and $\theta_{m+j} = 0$ $(j = 1, \cdots, m)$, we obtain $A = J^c = -J_c$.

The action is

$$S = \int_{t_0}^{t_1} \left(\theta_j(\boldsymbol{z}) \frac{dz^j}{dt} - H \right) dt. \tag{15}$$

By the variation

$$\delta_{\tilde{\boldsymbol{z}}(t)} S = \epsilon \int_a^b \left(A_{k\ell} \frac{dz^\ell}{dt} - \frac{\partial H}{\partial z^k} \right) \tilde{z}^k dt + O(\epsilon^2),$$

we obtain the Euler-Lagrange equation

$$A_{k\ell} \frac{dz^\ell}{dt} = \frac{\partial H}{\partial z^k}. \quad (k = 1, \cdots, n) \tag{16}$$

By multiplying $A^{-1} =: J$, we obtain

$$\frac{dz^\ell}{dt} = J^{k\ell} \frac{\partial H}{\partial z^k} \quad (k = 1, \cdots, n). \tag{17}$$

When, $n = 2m$, $\theta_j = p^j$ and $\theta_{m+j} = 0$ $(j = 1, \cdots, m)$, $J = A^{-1} = J_c$, thus (17) becomes a canonical equation (12).

One may generalize the action principle further by allowing the 2-form ω to be degenerate, i.e. the rank of the field tensor A may be less than the space dimension n, and, moreover, it may change as a function of $\boldsymbol{z}$ (the point where Rank(A) changes is a singularity; see Remark 1). Such system may not be transformed into an "explicit" differential equation such as (17). On the contrary, one may consider a general Hamilton's equation of the form of (17) with some antisymmetric $J(\boldsymbol{z})$ that may a smaller and non-constant rank. Such a Hamiltonian system may not be produced by an action principle. We call the latter a *non-canonical Hamiltonian system*, which will be the main subject of the preset lecture.

3 Non-canonical Hamiltonian mechanics

In this section, we consider a generalized Hamiltonian mechanics that are characterized by noncanonical Poisson operators. Here we consider only finite-dimensional systems, but the framework will be further generalized to infinite-dimensional systems in the next section.

3.1 Noncanonical Poisson operator and Casimir elements

Let X be a phase space of dimension n (an arbitrary finite number; we will generalize the theory for infinite-dimensional Hamiltonian system). Let $J(\boldsymbol{z}) \in \mathrm{End}(X)$ be an antisymmetric linear map (in general, varies as a regular function of $\boldsymbol{z}$ on X). [6] We consider a general Hamilton's equation of motion:

$$\frac{d}{dt}\boldsymbol{z} = J(\boldsymbol{z})\partial_{\boldsymbol{z}} H(\boldsymbol{z}). \tag{18}$$

We say that the Poisson operator $J(\boldsymbol{z})$ is *non-canonical*, if $\mathrm{Ker}(J(\boldsymbol{z}))$ has non-zero element. A function $C(\boldsymbol{z})$ ($\neq$ constant) that satisfies

$$J(\boldsymbol{z})\partial_{\boldsymbol{z}} C(\boldsymbol{z}) = 0 \tag{19}$$

is called a Casimir element. By the definition, $\partial_{\boldsymbol{z}} C(\boldsymbol{z}) \in \mathrm{Ker}(J(\boldsymbol{z}))$. However, a general $\boldsymbol{v} \in \mathrm{Ker}(J(\boldsymbol{z}))$ may not be "integrable" to produce a Casimir element $C(\boldsymbol{z})$ such that $\boldsymbol{v} = \partial_{\boldsymbol{z}} C(\boldsymbol{z})$. Integration (or *foliation*) is only possible under a limited condition:

Theorem 1 (Lie-Darboux) *Suppose that* $\mathrm{Rank}(J) = 2\nu < n$ *(the dimension of phase space), and* ν *is a constant independent to* $\boldsymbol{z}$*. Then, (19) has* $\mu = n - 2\nu)$ *independent solutions (Casimir elements)* $C_1(\boldsymbol{z}), C_2(\boldsymbol{z}), \cdots, C_\mu(\boldsymbol{z})$*. Choosing these Casimir elements as new coordinates, and also choosing other* 2ν *coordinates appropriately, we can transform variables as* $\boldsymbol{\zeta} = (\zeta_1, \cdots, \zeta_{2\nu}, C_{2\nu+1}, \cdots, C_{2\nu+\mu})$*, by which* J *can be transformed into a standard form*

$$J_s = \left(\begin{array}{cc|c} 0_\nu & I_\nu & \\ -I_\nu & 0_\nu & \\ \hline & & 0_\mu \end{array} \right). \tag{20}$$

Casimir elements are constants of motion. In fact,

$$\frac{d}{dt}C = -\{H, C\} = 0. \tag{21}$$

[6] Normally, we assume that the Poisson bracket defined by $\{a, b\} := (\partial_{\boldsymbol{z}} a, J\partial_{\boldsymbol{z}} b)$ satisfies Jacobi's identity (6). In a more general discussion, we may not require Jacobi's identity.

Notice that (21) holds for any arbitrary Hamiltonian H; the invariance of Casimir elements is due to the "topological defect" (kernel) of the Poisson operator J, but is not due a symmetry of the Hamiltonian (remember that a usual constant of motion is produced by a symmetry of a Hamiltonian).

By the fact that a Casimir element $C(\boldsymbol{z})$ (if it exists) is a constant of motion, every orbit is constrained on a level set of $C(\boldsymbol{z})$ that includes the initial point of the orbit. Such a manifold is called a *leaf*. We say that the phase space is *foliated* by Casimir leaves.

Example 1 (Nambu dynamics) *Let us consider a three-dimensional state vector* $\boldsymbol{P}$*, and a Poisson operator*

$$J = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & \Omega_1 & 0 \end{pmatrix}. \tag{22}$$

In terms of a three vector $\boldsymbol{\Omega} = {}^t(\Omega_1, \Omega_2, \Omega_3)$*, we may write* $J = -\boldsymbol{\Omega}\times$*. Evidently,* $C = (\boldsymbol{\Omega}, \boldsymbol{P})$ *is a Casimir element. The corresponding Poisson bracket may be written as*

$$\{A, B\} = (\partial_{\boldsymbol{P}} A, (\partial_{\boldsymbol{P}} B) \times (\partial_{\boldsymbol{P}} C)). \tag{23}$$

Notice that the right-hand side expression has an interesting symmetry; denoting it by a triple-term bracket $\{A, B, C\}$*, we observe* $\{A, B, C\} = \{B, C, A\} = \{C, A, B\}$*. Given a Hamiltonian* B*, the dynamics of an observable* A *is described by* $dA/dt = \{A, B, C\}$*, where* C *is a term determining the "geometry". The role of* B *(Hamiltonian) and* C *(Casimir element) may be switched by considering* C *is the Hamiltonian and* $-B$ *is the Casimir element. Nambu [6] proposed an interpretation that* C *is the second Hamiltonian.*

Remark 1 (Casimir foliation) *In general, Casimir leaves are not necessarily* symplectic leaves, *i.e., separating Casimir elements may not suffice to "canonicalize"* $J(\boldsymbol{z})$*. As Theorem 1 applies under rather limited conditions, a general topological defect* $\mathrm{Ker}(J)$ *may not determine a "global" foliation of the phase space. The point where* $\mathrm{Rank}(J(\boldsymbol{z}))$ *changes is singularity of the partial differential equation (19), which generates singular (*hyper-function*) Casimir elements [13].*

3.2 Energy-Casimir function

When we have a Casimir element $C(\boldsymbol{z})$ in a noncanonical Hamiltonian system, a transformation of the Hamiltonian $H_(\boldsymbol{z})$ such as (with an arbitrary

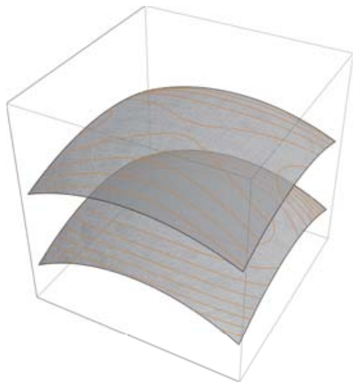

Figure 2: Low-dimensional cartoon of Casimir foliation of phase space. In a noncanonical Hamiltonian system, the dynamics is constrained on a level set (leaf) of Casimir invariant (the leaf on which the orbit is constrained in determined by the initial condition). In general, Casimir leaves have different curvatures than those of the energy shells (level sets of the energy norm), hence the effective energy dominating the dynamics constrained on Casimir leaves have rather complicated distributions, creating interesting dynamics, equilibria, or thermal equilibria.

real constant μ)

$$H(\boldsymbol{Z}) \mapsto H_\mu(\boldsymbol{z}) = H(\boldsymbol{z}) - \mu C(\boldsymbol{z}) \tag{24}$$

does not change the dynamics. In fact, Hamilton's equation (18) is invariant under this transformation:

$$\frac{d}{dt}\boldsymbol{z} = J\partial_{\boldsymbol{z}} H_\mu(\boldsymbol{z}) = J\partial_{\boldsymbol{z}} H(\boldsymbol{z}). \tag{25}$$

We call the transformed Hamiltonian $H_\mu(\boldsymbol{z})$ a *energy-Casimir* function.

Interpreting the parameter μ as a Lagrange multiplier of variational principle, $H_\mu(\boldsymbol{z})$ is the effective Hamiltonian with the constraint to restrict the Casimir element $C(\boldsymbol{z})$ to be a given value (since $C(\boldsymbol{z})$ is a constant of motion, its value is fixed by its initial value). As we will see in some examples, Hamiltonians are rather simple —they are often "norms" of the phase space.[7] However, an energy-Casimir functional may have a nontrivial structure. Geometrically, $H_\mu(\boldsymbol{z})$ is the distribution of $H(\boldsymbol{z})$ on a Casimir leaf

[7] In a "strongly coupled system", however, the Hamiltonian may be a nontrivial function (see Fig. 2). For example, remember the Ginzburg-Landau potential in a condensed spin system.

(hyper-surface of $C(\boldsymbol{z})$ = constant). If Casimir leaves are distorted with respect to the energy norm, the effective Hamiltonian may have complex distribution on the leaf.

3.3 Canonicalization

Suppose that a non-canonical Poisson operator J_{nc} is given in a standard form [8]

$$J_{nc} = \left(\begin{array}{c|c} J_c & \\ \hline & 0_\mu \end{array} \right), \tag{26}$$

where J_c is a $2\nu \times 2\nu$ symplectic matrix; see (20). There are two different methods to canonicalize (26).

3.3.1 Reduction method

When a noncanonical Poisson operator is casted in the standard form (26), $C_1(\boldsymbol{z}) = z^{2\nu+1}, \cdots, C_\mu = z^{2\nu+\mu}$ are apparent Casimir elements, and the Casimir leafs ($X_c := X/\mathrm{Ker}(J_{nc})$) are symplectic leaves. Hence, decomposing the null space $\mathrm{Ker}(J_{nc})$ from X, and restricting the dynamics on X_c, we obtain a canonicalized Poisson operator J_c.

3.3.2 Extension method

Adding μ new variables $\vartheta_1, \cdots, \vartheta_\mu$, we consider an extended state vector

$$\tilde{z} := (z^1, \cdots, z^n, \vartheta_1, \cdots, \vartheta_\mu) \in X^c := X \times \mathbb{R}^\mu.$$

We define a $2(\nu+\mu) \times 2(\nu+\mu)$ matrix such as

$$\widetilde{J} = \left(\begin{array}{c|cc} J_c & & \\ \hline & 0_\mu & I_\mu \\ & -I_\mu & 0_\mu \end{array} \right). \tag{27}$$

After reordering the variables, $\widetilde{J}$ becomes a $2(\nu+\mu) \times 2(\nu+\mu)$ symplectic matrix.

In this canonicalized system, the original Casimir elements $C_j = z^{2\nu+j}$ ($j = 1, \cdots, \mu$) are still constants of motion, but they are no longer Casimir

[8]Note that an arbitrary noncanonical Poisson operator my not be transformed into this standard form; see Remark 1.

elements of the extended system —canonical system does not have any non-trivial Casimir elements. The constancy of $C_j(\boldsymbol{z})$ is, now, due to the symmetry of the Hamiltonian; the newly introduced variables ϑ_j is, of course, absent in the Hamiltonian $H(\boldsymbol{z})$ of the original system. Hence,

$$\frac{d}{dt}z^{2\nu+j} = -\partial_{\vartheta_j} H = 0.$$

4 Foliation by adiabatic invariants

Phase space foliation provided by adiabatic invariants is shown to impart simultaneous long-scale order and short-scale disorder to a Hamiltonian system. A plasma confined in a magnetosphere is invoked for unveiling the organizing principle —in the vicinity of a magnetic dipole, the plasma self-organizes to a state with a steep density gradient. The resulting nontrivial structure has maximum entropy in an appropriate, constrained phase space. One could view such a phase space as a macroscopic leaf (separating microscopic action-angle variables) of the scale hierarchy that is foliated in terms of Casimir invariants.

4.1 Microscopic description of charged particle dynamics

The Hamiltonian of a charged particle is a sum of the kinetic energy and the potential energy:

$$H = \frac{m}{2}v^2 + q\phi, \tag{28}$$

where $\boldsymbol{v} := (\boldsymbol{P} - q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential, m (q) is the particle mass (charge). In the present work, we may treat electrons and ions equally (in later discussion, we will neglect ϕ assuming charge neutrality, but generalization to a non-neutral plasma will be interesting [17]). Denoting by $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2}v_\perp^2 + \frac{m}{2}v_\parallel^2 + q\phi. \tag{29}$$

The velocities are related to the mechanical momentum as $\boldsymbol{p} := m\boldsymbol{v}$, $\boldsymbol{p}_\parallel := m\boldsymbol{v}_\parallel$, and $\boldsymbol{p}_\perp := m\boldsymbol{v}_\perp$.

4.2 Creation of an action-angle pair by magnetization

In a strong magnetic field, $\boldsymbol{v}_\perp$ can be decomposed into a small-scale cyclotron motion $\boldsymbol{v}_c$ and a macroscopic guiding-center drift motion $\boldsymbol{v}_d$. The periodic cyclotron motion $\boldsymbol{v}_c$ can be "quantized" to write

$$\frac{m}{2}v_c^2 = \mu\omega_c(\boldsymbol{x})$$

in terms of the magnetic moment μ and the cyclotron frequency $\omega_c(\boldsymbol{x})$; the adiabatic invariant μ and the gyration phase $\vartheta_c := \omega_c t$ constitute an action-angle pair. In the standard interpretation, in analogy with the Landau levels in quantum theory, ω_c is the *energy level* and μ is the *number* of quasi-particles (quantized periodic motions) at the corresponding energy level.

The macroscopic part of the perpendicular kinetic energy is expressed as

$$\frac{m}{2}v_d^2 = (P_\theta - q\psi)^2/(2mr^2),$$

where P_θ is the angular momentum in the θ direction and r is the radius from the geometric axis. In terms of the canonical-variable set

$$\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_{||}, \theta, P_\theta),$$

the Hamiltonian of the guiding center (or, the quasi-particle) becomes

$$H_c = \mu\omega_c + \frac{1}{2m}p_{||}^2 + \frac{1}{2m}\frac{(P_\theta - q\psi)^2}{r^2} + q\phi. \tag{30}$$

Note that the energy of the cyclotron motion has been quantized in term of the frequency $\omega_c(\boldsymbol{x})$ and the action μ; the gyro-phase ϑ_c has been coarse grained (integrated to yield 2π).

4.3 Boltzmann distribution

The standard Boltzmann distribution function is derived when we assume that d^3vd^3x is an invariant measure and the Hamiltonian H is the determinant of the ensemble. Maximizing the entropy $S = -\int f \log f d^3vd^3x$ keeping the total energy $E = \int Hfd^3vd^3x$ and the total particle number $N = \int fd^3vd^3x$ constant, we obtain

$$f(\boldsymbol{x}, \boldsymbol{v}) = Z^{-1}e^{-\beta H}, \tag{31}$$

where Z is the normalization factor ($\log Z - 1$ is the Lagrange multiplier on N) and β is the inverse temperature (the Lagrange multiplier on E). The corresponding configuration-space density,

$$\rho(\boldsymbol{x}) = \int fd^3v \propto e^{-\beta q\phi}, \tag{32}$$

becomes constant for a charge neutral system ($\phi = 0$).

Needless to say that the Boltzmann distribution or the corresponding configuration-space density, with an appropriate Jacobian multiplication, is independent of the choice of phase-space coordinates. Moreover, the density is invariant no matter whether we quantize the cyclotron morion or not. Let us confirm this fact by a direct calculation. For the Boltzmann distribution of the "guiding-center plasma"

$$
\begin{aligned}
f(\mu, v_d, v_\parallel; \boldsymbol{x}) &= Z^{-1} e^{-\beta H_c} \\
&= Z^{-1} e^{-\beta\left(\mu\omega_c(\boldsymbol{x}) + m v_d^2/2 + m v_\parallel^2/2 + q\phi(\boldsymbol{x})\right)},
\end{aligned}
\tag{33}
$$

the density is given by

$$
\rho(\boldsymbol{x}) = \int f d^3 v = \int f \, \frac{2\pi\omega_c}{m} d\mu dv_d dv_\parallel \propto e^{-\beta q \phi},
\tag{34}
$$

exactly reproducing (4).

4.4 Equilibrium on macroscopic hierarchy

Now we formulate the "macroscopic hierarchy" on which the thermal equilibrium creates a structure. The adiabatic invariance of the magnetic moment μ (the *number* of the quantized quasi-particles) imposes a *topological constraint* on the motion of particles; it is this constraint that is the root-cause of a macroscopic hierarchy and of structure formation. Mathematically, the *scale hierarchy* is equivalent to a foliation of the phase space. To explain how the scale hierarchy is formulated, we start by the general (micro-macro total) formulation, and then separate the microscopic action-angle pair μ-ϑ_c; the *macroscopic phase space* is the remaining sub-manifold immersed in the general phase space, which we delineate as a leaf of the foliation in terms of a *Casimir invariant* —if there is a nontrivial function C satisfying $\{G, C\} = 0$ for every G, we say that the Poisson bracket $\{ \ , \ \}$ is *non-canonical*, and call C a *Casimir invariant*; see Sec. 3.

The Poisson bracket on the total phase space, spanned by the canonical variables $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta)$, is

$$
\{F, G\} := (\partial_{\boldsymbol{z}} F, \mathcal{J} \partial_{\boldsymbol{z}} G),
$$

where $(\boldsymbol{u}, \boldsymbol{v}) := \int u_j v^j d^6 z$ is the inner-product and J is the canonical symplectic matrix:

$$
\mathcal{J} := \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}, \quad J_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\tag{35}
$$

The equation of motion for the Hamiltonian H_c is written as $dz^j/dt = \{z^j, H_c\}$. Notice that the quantization of the cyclotron motion suppresses change in μ. Liouville's theorem determines the invariant measure d^6z, by which we obtain the Boltzmann distribution (33).

To extract the macroscopic hierarchy, we "separate" the microscopic variables (ϑ_c, μ) by modifying the symplectic matrix as

$$J_{nc} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}. \tag{36}$$

The Poisson bracket

$$\{F, G\}_{nc} := (\partial_{\boldsymbol{z}} F, \mathcal{J}_{nc} \partial_{\boldsymbol{z}} G)$$

determines the kinematics on the macroscopic hierarchy; the corresponding kinetic equation $\partial_t f + \{H_c, f\}_{nc} = 0$ reproduces the familiar drift-kinetic equation.

The nullity of J_{nc} makes the Poisson bracket $\{\,,\,\}_{nc}$ *non-canonical* [5]. Evidently, μ is a Casimir invariant (more generally $C = g(\mu)$ with g being any smooth function). The level-set of μ, a leaf of the Casimir foliation, identifies what we may call the *macroscopic hierarchy.* By applying Liouville's theorem to the Poisson bracket $\{\,,\,\}_{nc}$, the invariant measure on the macroscopic hierarchy is $d^4z = d^6z/(2\pi d\mu)$, the the total phase-space measure modulo the microscopic measure. The most probable state (statistical equilibrium) on the macroscopic ensemble must maximize the entropy with respect to this invariant measure. The variational principle is set up following the standard procedure —immersing the macroscopic hierarchy into the general phase space, and incorporating the constraints through the Lagrange multipliers: We maximize entropy $S = -\int f \log f \, d^6z$ for a given particle number $N = \int f d^6z$, a quasi-particle number $M = \int \mu f d^6z$, and an energy $E = \int H_c f d^6z$, to obtain the distribution function

$$f = f_\alpha := Z^{-1} e^{-(\beta H_c + \alpha\mu)}, \tag{37}$$

where α, β, and $\log Z - 1$ are, respectively the Lagrange multipliers on M, E, and N. In this "grand-canonical" distribution function, α/β is the chemical potential associated with the quasi-particles.

We can also derive (37) by an *energy-Casimir function.* With a Casimir element μ, we can transform the Hamiltonian as $H_c \mapsto H_\alpha := H_c + \alpha\mu$ (α is an arbitrary constant) without changing the macroscopic dynamics; H_α is called an energy-Casimir function [5]. The Boltzmann distribution with respect to H_α is equivalent to (37).

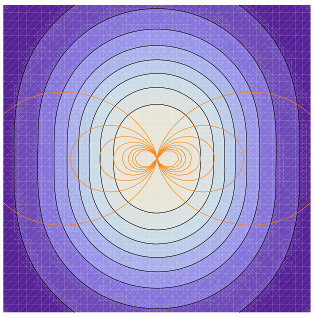

Figure 3: Density distribution (contours) and the magnetic field lines (level-sets of ψ; orange lines) in the neighborhood of a point dipole.

The factor $e^{-\alpha\mu}$ in f_α yields a direct ω_c dependence of the configuration-space density:

$$\rho = \int f_\alpha \frac{2\pi\omega_c}{m} d\mu dv_d dv_\parallel \propto \frac{\omega_c(\boldsymbol{x})}{\beta\omega_c(\boldsymbol{x}) + \alpha}, \tag{38}$$

which may be compared with the density (34) evaluated for the Boltzmann distribution ($\phi = 0$ assuming charge neutrality). Notice that the Jacobian $(2\pi\omega_c/m)d\mu$ multiplying the macroscopic measure d^4z reflects the distortion of the macroscopic phase space (Casimir leaf) caused by the magnetic field. Figure 3 shows the density distribution and the magnetic field lines.

We now see that the self-organized confinement of plasma by a dipole magnetic field is due to the phase-space foliation by the adiabatic invariant = Casimir element of the noncanonicalized Poisson operator; the effective phase space on which the thermal equilibrium is achieved is a Casimir leaf, which is the "macroscopic scale hierarchy" coarse-graining the microscopic angle variable.

5 Infinite-dimensional Hamiltonian mechanics

From this section, we consider systems where "states" are represented by functions of space-time. Quantum mechanics will be discussed as an example of canonical Hamiltonian system on a function space. Fluid/plasma mechanics are noncanonical systems. Their Poisson operators are "inhomogeneous" on the phase space (function space) —their inhomogeneities determine the "nonlinearities" of fluid/plasma systems.

5.1 Function space

A *state* that is represented by some function $u(\boldsymbol{x})$ (possibly having multiple components such as vector or spinor functions) are recognized as a "point" on a function space; $\boldsymbol{x}$ is a point of a domain Ω that is an open set of $\mathbb{R}^n$; $u(\boldsymbol{x})$ is a point of a function space that we denote $\mathcal{X}$. Endowing a functions space $\mathcal{X}$ with the notion of *inner product* (we will denote an inner product by $\langle u, v\rangle$), we may consider a "basis" to span the function, which turns out to be a system of (countably) infinite number of functions, [9] thus we may regard $\mathcal{X}$ as an infinite-dimensional vector space, i.e. we may write

$$u = \sum_{j=1}^{\infty} \langle u, \varphi_j \rangle \varphi_j \quad (\forall u \in \mathcal{X}),$$

with an orthonormal basis $\{\varphi_1, \varphi_2, \cdots\}$. A complete normed space with its norm $\|u\|$ given by an inner product as $\|u\| = \sqrt{\langle u, u\rangle}$ is called a *Hilbert space*. [10]

[9] Usually, we assume that the first axiom of separability holds for the space of functions, and then, the potency of the space must be at most "countably" infinite. This might be somewhat surprising. Since an interval Ω on a real axis $\mathbb{R}$, for example, is a continuously infinite set, a function $f(x)$ on Ω might be thought to have a continuously infinite degree of freedom. However, this is not true. For example, let us consider a space of continuous functions; $C^0(\Omega)$. Then, a function $f(x) \in C^0(\Omega)$ is already determined uniquely when its values $f(q_k)$ are specified at every $q_k \in \Omega \cap \mathbb{Q}$ ($\mathbb{Q}$ is the totality of rational numbers, which is a countable set), because there is a sequence $q_j \to x$ for every $x \in \Omega$ by which $f(x) = \lim_{j\to\infty} f(q_j)$ is uniquely determined by the continuity of $C^0(\Omega)$.

[10] A complete normed space $\mathcal{B}$ is called a *Banach space* (by *complete*, we mean that every Cauchy sequence converges to a point in the space). We call a map $\mathcal{B} \to \mathbb{K}$ (field of scalar, which is either $\mathbb{R}$ or $\mathbb{C}$) a *functional.* The linear space of bounded linear functionals on a Banach space $\mathcal{B}$ is called the *dual space* of $\mathcal{B}$, which we denote by $\mathcal{B}^*$. For a Hilbert space $\mathcal{X}$, Riesz' theorem says that $\mathcal{X}^* = \mathcal{X}$, i.e. every bounded linear functional $F(u)$ of $u \in \mathcal{X}$ can be written as $F(u) = \langle f, u\rangle$ with some $f \in \mathcal{X}$.

Example 2 (Lebesgue space $L^2(\Omega)$ and Fourier expansion) *Let Ω be a connected open set of $\mathbb{R}^n$. We consider multi-component complex functions such as $u(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \cdots, u_m(\boldsymbol{x}))$, and define an inner product by*

$$\langle u, v\rangle = \int_\Omega \sum_{j=1}^{m} \overline{u_j(\boldsymbol{x})} v_j(\boldsymbol{x}) \, d^n x, \tag{39}$$

where $d^n x$ is the n-dimensional Lebesgue measure. By the property of Lebesgue integrals, the space $L^2(\Omega)$ of Lebesgue-measurable functions endowed with the norm $\|u\| = \sqrt{\langle u, u\rangle}$ is a Hilbert space.

For example, suppose that $\Omega = (0,1) \subset \mathbb{R}$. With an orthonormal basis $\{\varphi_1, \varphi_2, \cdots\}$ with $\varphi_k = \sqrt{2}\sin(k\pi x)$, we may Fourier-expand every $u \in L^2(0,1)$:

$$u = \sum_{j=1}^{\infty} \langle u, \varphi_j\rangle \varphi_j, \tag{40}$$

where the precise meaning of the convergence of the infinite sum is that

$$\lim_{\nu\to\infty} \left\| u - \sum_{j=1}^{\nu} \langle u, \varphi_j\rangle \varphi_j \right\| = 0.$$

When a state u is dynamical, its orbit is represented by a function of space-time $u(\boldsymbol{x}, t)$. Regarding that u is a point on a function space $\mathcal{X}$ and that $\mathcal{X}$ is an infinite-dimensional vector space, we often write an orbit as $u(t)$ (just as we denote by $\boldsymbol{z}(t)$ an orbit of a state vector $\boldsymbol{z} \in \mathbb{R}^n$). The temporal derivative of $u(t)$ may be defined by the limit with respect to the norm of $\mathcal{X}$ as $\dot{u}(t) \in \mathcal{X}$ such that

$$\lim_{\epsilon\to 0} \left\| \frac{u(t+\epsilon) - u(t)}{\epsilon} - \dot{u}(t) \right\| = 0.$$

If we represent $u(t)$ by its components as $u(t) = \sum_j u_j(t)\varphi_j$ (cf. the Fourier expansion (40) in Example 2), we may write $\dot{u} = \sum_j [du_j(t)/dt]\varphi_j$. Hence, $\dot{u}$ might be denoted by $du(t)/dt$ as we write $d\boldsymbol{z}(t)/dt$ for a finite-dimensional state vector $\boldsymbol{z}(t)$. However, to avoid confusion with "convective (or total) derivatives", we will denote a temporal derivative as a partial differential $\partial_t u$ (needless to say, for regular functions $u(\boldsymbol{x}, t)$, $\dot{u}(t)|_{\boldsymbol{x}} = \partial_t u(\boldsymbol{x}, t)$ for every $\boldsymbol{x} \in \Omega$ and t).

5.2 Infinite-dimensional Hamiltonian system

We are going to generalize Hamiltonian formalism (18) to an infinite-dimensional systems; a general Hamilton's equation of motion will be written as

$$\partial_t u = \mathcal{J}\partial_u H(u), \tag{41}$$

where $u(t)$ is an orbit in a Hilbert space $\mathcal{X}$, $H(u)$ is a Hamiltonian that is a smooth functional on $\mathcal{X}$, ∂_u is the *gradient* in $\mathcal{X}$, and $\mathcal{J}$ is an antisymmetric linear map $\mathcal{X} \to \mathcal{X}$.

We have yet to define what the gradient is in a Hilbert space. Remember that the gradient $\partial_{\boldsymbol{z}} f$ of a smooth function $f(\boldsymbol{z})$ on a Euclidean space $\mathbb{R}^n$ is a vector such that (invoking a small real number ϵ)

$$\delta f(\boldsymbol{z}) = f(\boldsymbol{z} + \epsilon\boldsymbol{\delta}) - f(\boldsymbol{x}) = \epsilon(\partial_{\boldsymbol{z}} f, \boldsymbol{\delta}) + O(\epsilon^2) \quad (\forall \boldsymbol{\delta} \in \mathbb{R}^n).$$

Generalizing to a Hilbert space $\mathcal{X}$ (generally on a field of scalar $\mathbb{C}$), we may define the $\partial_u F(u)$ of a smooth real functional $F(u)$, [11] we define

$$\delta F(u) = F(u + \epsilon\delta) - F(u) = \epsilon \Re \langle \partial_u F(u), \delta\rangle + O(\epsilon^2) \quad (\forall \delta \in \mathcal{X}), \tag{42}$$

where $\langle a, b\rangle$ is the inner product of $\mathcal{X}$. [12]

Example 3 *Let $f(\xi)$ be a smooth function on $\mathbb{R}$, and consider a functional $F(u) = \int_\Omega f(u)\, d^n x$ for $u \in L^2(\Omega)$. If $\Omega \in \mathbb{R}^n$ is a bounded domain, $F(u)$ is a smooth functional on $L^2(\Omega)$. We easily verify*

$$\partial_u F(u) = f'(u), \tag{43}$$

where $f'(\xi) = df(\xi)/d\xi$.

5.3 Schrödinger's equation

As an example of canonical Hamiltonian system on a function space, let us consider a Schrödinger equation:

$$\partial_t \psi = \frac{1}{\sqrt{-1}\hbar}\mathcal{H}\psi, \tag{44}$$

[11] We often encounter functionals that are not smooth (for example, including differential operators). Then, we have to limit variations δ to some appropriate subset of $\mathcal{X}$, and weaken (generalize) the meaning of derivatives; cf. [13].

[12] If we consider a general Banach space $\mathcal{B}$, $\langle \partial_u F(u), \circ\rangle$ may be regarded as a linear functional on $\mathcal{B}$, and $\partial_u F(u) \in \mathcal{B}^*$. For some Banach spaces (such as $L^p(\Omega)$ with $1 < p < \infty$), the dual spaces are specified ($(L^p(\Omega))^* = L^q(\Omega)$, if $p^{-1} + q^{-1} = 1$), thus the generalization of the notion of gradients to such Banach spaces is straightforward.

where $\psi \in L^2(\mathbb{R}^3)$ is a "wave function" (here we consider a scalar function), and $\mathcal{H}$ is a quantized Hamiltonian which is obtained by replacing the momentum $\boldsymbol{P}$ of classical mechanics by a differential operator $-\hbar\sqrt{-1}\nabla$; for a particle moving in a scalar potential $\phi(\boldsymbol{x})$,

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + \phi(\boldsymbol{x}). \tag{45}$$

We define a real functional by

$$H(\psi) = \langle \psi, \mathcal{H}\psi \rangle, \tag{46}$$

which evaluates the expectation value of the energy=Hamiltonian $\mathcal{H}$. [13]

By the Hermitian property of $mathcal H$, we find

$$\partial_\psi H(\psi) = 2\mathcal{H}\psi. \tag{47}$$

[Q] For functions ψ and φ such that $\psi, \varphi|_{|\boldsymbol{x}|^2\to\infty} = 0$, show that $\langle \psi, \mathcal{H}\phi \rangle = \langle \psi, \mathcal{H}\phi \rangle$, i.e. $\mathcal{H}$ is a Hermitian operator. By this fact, show that $H(\psi)$ is a real functional. Also, derive (47).

To cast Schrödinger's equation into a Hamiltonian form (41), the Poisson operator must be

$$\mathcal{J} = \frac{-\sqrt{-1}}{2\hbar},$$

which is evidently regular ($\mathrm{Ker}(\mathcal{J}) = \{0\}$).

5.4 Fluid equation

Here we introduce an example of infinite-dimensional noncanonical Hamiltonian mechanics from fluid mechanics theory. Let us consider a two-dimensional incompressible ideal flow contained in a bounded domain $\Omega \subset \mathbb{R}^2$. [14] We denote by $\boldsymbol{P}$ the momentum of the fluid, which is assumed to be a smooth function on Ω. By the incompressibility condition ($\nabla \cdot \boldsymbol{P} = 0$), we may write $\boldsymbol{P} =^t (\partial_y \psi, -\partial_x \psi)$ with a scalar function $\psi(x, y)$. By the

[13] Here we define the functional $H(\psi)$ on a subspace $H^2(\mathbb{R}^3) = \{\psi;\ \|psi\| < \infty, \|\nabla\psi\| < \infty, \|\nabla^2\psi\| < \infty\}$, which is a sens subset of $L^2(\mathbb{R}^3)$.

[14] For the derivation of the vorticity equation, see Sec. subsubsec:Euler. A vorticity is the exterior derivative of a momentum. A momentum is a differential 1-form, thus a vorticity is a 2-form. In two-dimensional space, 2-form has a single component; see Sec. A.3.2 of Appendix A.

boundary condition $\boldsymbol{n}\cdot\boldsymbol{P}=0$, we may assume $\psi=0$ on the boundary $\partial\Omega$. Let ω be a *vorticity* of $\boldsymbol{P}$, which is related to ψ by

$$\left\{\begin{array}{l} -\nabla^2\psi=\omega \quad (\text{in }\Omega), \\ \psi=0 \quad (\text{on }\partial\Omega). \end{array}\right. \tag{48}$$

Solving the Poisson equation (48), we may write

$$\omega=\mathcal{K}\psi. \tag{49}$$

The operator $\mathcal{K}$ is a Hermitian operator from $L^2(\Omega)$ to $H^2(\Omega)$. Choosing $\omega\in L^2(\Omega)$ to be the state vector, the fluid energy reads

$$H(\omega)=\frac{1}{2m}\|\boldsymbol{P}\|^2=\frac{1}{2m}\langle\omega,\mathcal{K}\omega\rangle. \tag{50}$$

We observe

$$\partial_\omega H(\omega)=\mathcal{K}\omega. \tag{51}$$

Denoting

$$\{a,b\}=(\partial_x a)(\partial_y b)-(\partial_y a)(\partial_x b),$$

we define a Poisson operator

$$\mathcal{J}=\{\omega,\circ\}. \tag{52}$$

The Hamilton's equation (41) reads

$$\partial_t\omega=\frac{1}{m}\{\omega,\mathcal{K}\omega\}=-\boldsymbol{V}\cdot\nabla\omega, \tag{53}$$

where $\boldsymbol{V}=\boldsymbol{P}/m$ is the fluid velocity.

[Q] Derive (53), and show that the two-dimensional version of (62) reduces into (53) under the barotropic condition.

The Poisson operator of (52) is an inhomogeneous (depending on the state variable ω) differential operator. It has infinitely many Casimir elements; for every smooth function $f(\xi)$,

$$C_f(\omega)=\int_\Omega f(\omega)\,d^2x$$

is an Casimir element.

6 Hamiltonian mechanics of fluids and plasmas

Ideal (energy-conserving) mechanics of a fluid or a plasma is described by a noncanonical Hamilton's equation of motion on a function space. Analyzing the foliation of the phase space, we delineate nontrivial structures that are self-organized on Casimir leaves.

6.1 Naïve forms of fluid/plasma equations

6.1.1 Euler equation and vorticity equation of ideal incompressible fluid

Let Ω be a fixed domain in $\mathbb{R}^n$ ($n = 2$ or 3) whose boundary $\partial\Omega$ (if it exists) is a smooth surface.

We denote by $\boldsymbol{P}$ (n-dimensional vector field (in fact, 1-form) defined on Ω) a fluid momentum ($\boldsymbol{P} = m\boldsymbol{V}$ with a fluid velocity $\boldsymbol{V}$ and particle mass m), and by p (scalar function on Ω) a pressure. We assume that the fluid is incompressible, and has a constant density which we normalize to unity. The *Euler equation* describes the ideal (viscosity = 0) fluid motion:

$$\partial_t \boldsymbol{P} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{P} = -\nabla p \quad (\text{in } \Omega)\,, \tag{54}$$

$$\nabla\cdot\boldsymbol{P} = 0 \quad (\text{in } \Omega)\,, \tag{55}$$

$$\boldsymbol{n}\cdot\boldsymbol{P} = 0 \quad (\text{on } \partial\Omega)\,. \tag{56}$$

We may rewrite (54) as

$$\partial_t \boldsymbol{P} - \boldsymbol{V}\times(\nabla\times\boldsymbol{P}) = -\nabla\tilde{p} \quad (\text{in } \Omega)\,, \tag{57}$$

where $\tilde{p} = p + P^2/2m$ is the total enthalpy of the fluid.

Operating curl on the both-hand sides of (57), we obtain the *vorticity equation*:

$$\partial_t \boldsymbol{\omega} - \nabla\times(\boldsymbol{V}\times\boldsymbol{\omega}) = 0 \quad (\text{in } \Omega), \tag{58}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{P}$ is the *vorticity*.

6.1.2 Compressible ideal fluid

In an incompressible fluid, the pressure p is determined to let $\boldsymbol{V}$ to satisfy the incompressible condition (55) and the boundary condition (56). Physically, however, p must be determined by some equation of state in its relation to the density ρ (we denote by ρ the number density, and by $\rho_m = m\rho$ the mass density,where m is the particle mass) and temperature T (or entropy S). [15]

The density ρ must obey the particle conservation law: [16]

$$\partial_t \rho + \nabla \cdot (\boldsymbol{V} \rho) = 0. \tag{59}$$

Including a non-constant density ρ, the momentum equation (54) becomes [17]

$$\partial_t \boldsymbol{P} - \boldsymbol{V} \times (\nabla \times \boldsymbol{P}) = -\frac{1}{\rho} \nabla p - \nabla \left(\frac{P^2}{2m} \right) \quad (\text{in } \Omega). \tag{60}$$

In a *barotropic model*, we assume $\rho^{-1}(\nabla p) = \nabla h(\rho)$ with a molar enthalpy $h(\rho)$. Since the right-hand side of (60) is an exact differential, the vorticity equation (58) remains the same.

When we consider a more general equation of state, we invoke σ to write $p = p(\rho, \sigma)$ and $(\rho)^{-1}(\nabla p) = \nabla h(\rho) - T \nabla \sigma$ In an ideal fluid, σ is conserved along each streamline: [18]

$$\partial_t \sigma + \boldsymbol{V} \cdot \nabla \sigma = 0 \quad (\text{in } \Omega). \tag{61}$$

In general, the right-hand side of (60) is not an exact differential, and the vorticity equation (58) is modified as

$$\partial_t \boldsymbol{\omega} - \nabla \times (\boldsymbol{V} \times \boldsymbol{\omega}) = \nabla T \times \nabla \sigma. \tag{62}$$

The right-hand side of (62) represents the *baroclinic effect*, by which vorticity can be generated.

6.1.3 Charged fluid (plasma)

In a charged fluid (plasma), the momentum and energy (Hamiltonian) include the electromagnetic (EM) terms. Denoting the four potentials by

[15] Incompressibility corresponds to an infinite sound velocity. Hence, for a slow motion of fluid with respect to the propagation of sound wave may be described by an incompressible model.

[16] A density ρ is an n-form. In terms of Lie derivative of n-form ($L_{\boldsymbol{V}} \rho = \nabla \cdot (\boldsymbol{V} \rho)$), (59) means $\partial_t \rho + L_{\boldsymbol{V}} \rho = 0$.

[17] The momentum $\boldsymbol{P}$ is a 1-form. In terms of the Lie derivative of 1-form ($L_{\boldsymbol{V}} \boldsymbol{P} = -\boldsymbol{V} \times (\nabla \times \boldsymbol{P}) + \nabla(\boldsymbol{V} \cdot \boldsymbol{P})$), (60) means $\partial_t \boldsymbol{P} + L_{\boldsymbol{V}} \boldsymbol{P} + \rho^{-1} dp = dE$, where the left-hand side is the variation of momentum and mechanical work of compression, while the right-hand side is the variation of mechanical energy $E = P^2/2m$.

[18] Here we consider that σ is a 0-form (scalar). In terms of Lie derivative of 0-form ($L_{\boldsymbol{V}} \sigma = \boldsymbol{V} \cdot \nabla \sigma$), (59) means $\partial_t \sigma + \mathcal{L}_V \sigma = 0$.

$(\phi, \boldsymbol{A})$, and the charge by q, we transform

$$\boldsymbol{P} = m\boldsymbol{V} \quad \mapsto \quad \boldsymbol{P} = m\boldsymbol{V} + q\boldsymbol{A}, \tag{63}$$

$$E = \frac{mV^2}{2} \quad \mapsto \quad E = \frac{mV^2}{2} + q\phi. \tag{64}$$

With these "canonical" momentum and energy, (60) describes the dynamics of charged fluid: denoting $\Phi = mV^2/2 + h + q\phi$, we may write the canonical momentum equation as

$$\partial_t \boldsymbol{P} - \boldsymbol{V} \times (\nabla \times \boldsymbol{P}) = -\nabla\Phi + T\nabla\sigma \quad (\text{in } \Omega). \tag{65}$$

Combining with the mass conservation law (59) and entropy conservation law (61), as well as the boundary condition (56), we obtain a system of charged fluid equations; to determine the EM potentials $(\phi, \boldsymbol{A})$, we must solve Maxwell's equations with the four currents $(q\rho, q\rho\boldsymbol{V})$ simultaneously.

Usually, a plasma consists of different species of charged particles (typically ions and electrons with the total charge approximately canceled), Then, we consider a multi-species fluid defining macroscopic quantities $\rho, \boldsymbol{P}, h, T, \sigma$ for each component.

6.1.4 Magnetohydrodynamics (MHD) model

Instead of considering a plasma as a multi-species charged fluid, we may formulate a reduced macroscopic equation by just considering that the fluid can carry a current and receives a macroscopic Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$, where $\boldsymbol{J}$ is a macroscopic current that is related with magnetic field $\boldsymbol{B}$ by (denoting by μ_0 the vacuum permeability)

$$\nabla \times \boldsymbol{B} = \boldsymbol{J}/\mu_0.$$

Here we have neglected the displacement current $c^{-2}\partial_t \boldsymbol{E}$ of Maxwell's equation. Combining with Faraday's law,

$$\left\{ \begin{array}{l} \partial_t \boldsymbol{V} - \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) = \frac{1}{m\rho} \left[-\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}/\mu_0 \right], \\ \partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \end{array} \right. \tag{66}$$

Coupling with the mass conservation law (59) (or, the incompressibility condition $\nabla \cdot \boldsymbol{V} = 0$), we obtain as closed system (we do not need to invoke other relations of Maxwell's equation).

6.2 Hamiltonian formalism

The foregoing fluid/plasma models are free from dissipations of energy, so we may cast them in (generalized) Hamiltonian forms.

6.2.1 Compressible fluid/plasma models

For simplicity we consider a neutral fluid governed by (59)-(60); generalization to charged fluids is straightforward by the relations (63) and (64). We also assume barotropic fluid $h = h(\rho)$. The state vector is a four-component function $\boldsymbol{u} = {}^t(\rho, \boldsymbol{P})$. We consider the entire space $\Omega = \mathbb{R}^3$ and assume that $\boldsymbol{u}$ vanishes at infinity to avoid the complication by boundary conditions.. The Hamiltonian is the sum of the fluid kinetic energy and the internal energy (we denote by $\mathcal{E}$ the molar internal energy, which is a function of only ρ in the barotropic model; $d(\rho\mathcal{E})/d\rho = h = \mathcal{E} + p/\rho$)

$$H(u) = \int \left[\frac{P^2}{2m} + \mathcal{E}(\rho) \right] \rho d^3x. \tag{67}$$

Then, we must define

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{P})\times \end{pmatrix} \tag{68}$$

to reproduce the fluid equations (59)-(60). Notice that the element $-\rho^{-1}(\nabla \times \boldsymbol{P})\times$ in the operator $\mathcal{J}$ is the generalization of (22) by $\boldsymbol{\Omega} \mapsto \rho^{-1}(\nabla \times \boldsymbol{P})$; the latter is a function of space.

[Q] Show that the Poisson operator of (68) is anti-symmetric. Here the inner product of state vectors is defined as $\langle u, v \rangle = \int_\Omega u \cdot v \; d^3x$.

6.2.2 MHD model

The state vector is $\boldsymbol{u} = {}^t(\rho, m\boldsymbol{V}, \boldsymbol{B})$. [19] Putting

$$\begin{aligned} H &= \int \left\{ \left[\frac{mV^2}{2} + \mathcal{E}(\rho) \right] \rho + \frac{B^2}{2\mu_0} \right\} d^3x. && (69) \\ \mathcal{J} &= \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -(m/\rho)(\nabla \times \boldsymbol{V})\times & (\nabla \times \circ) \times \boldsymbol{B}/\rho \\ 0 & \nabla \times [\circ \times \boldsymbol{B}/\rho] & 0 \end{pmatrix} && (70) \end{aligned}$$

we may write the MHD equations (59)-(66) in a Hamiltonian form (18).

[Q] Show that the Poisson operator of (70) is anti-symmetric.

[19] If the domain Ω has a boundary $\partial\Omega$, we have to give boundary conditions. Usually, we assume perfectly conducting boundary, and impose $\boldsymbol{n} \cdot \boldsymbol{V} = 0$, $\boldsymbol{n} \cdot \boldsymbol{B} = 0$. When Ω is multiply connected, however, we need also give a *flux condition* to fix the harmonic (vacuum) magnetic field. Decomposing $\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{B}_H$ ($\boldsymbol{B}_\Sigma \in L^2_\Sigma(\Omega)$, $\boldsymbol{B}_H \in L^2_H(\Omega)$), the harmonic part $\boldsymbol{B}_H$ is fixed by the perfectly conducting boundary, thus the dynamical part is only $\boldsymbol{B}_\Sigma$ (see Theorem 4); the state vector is $u = (\rho, m\boldsymbol{V}, \boldsymbol{B}_\Sigma)$.

6.2.3 Incompressible models

To formulate incompressible models (cf. Sec. 6.1.1), we need the orthogonal decomposition of the Hilbert space $L^2_\sigma(\Omega)$ of incompressible vector fields and the orthogonal projection $\mathcal{P}_\sigma$ onto $L^2_\sigma(\Omega)$; see Appendix B.

In an incompressible system, the pressure term ∇p in the momentum equation must be deemed as a "correction term" to guarantee the incompressibility condition $\nabla \cdot \boldsymbol{P} = 0$. It is noting but the effect of the projector $\mathcal{P}_\sigma$, i.e. for an arbitrary $\boldsymbol{u} \in L^2(\Omega)$, $\mathcal{P}_\sigma \boldsymbol{u} = \boldsymbol{u} - \nabla\varphi$ with some $\nabla\varphi \in L^2(\Omega)/L^2_\sigma(\Omega)$. [20]

Applying the projector $\mathcal{P}_\sigma$ on the both sides of (54), we obtain

$$\partial_t \boldsymbol{V} = -\mathcal{P}(\nabla \times \boldsymbol{V}) \times \boldsymbol{V}. \tag{71}$$

The Hamiltonian of the incompressible Euler fluid (state vector is $u = \boldsymbol{P}$ is obtained by omitting the internal energy $\mathcal{E}(\rho)$ (which becomes a constant when $\rho = 1$) in (67):

$$H = \frac{1}{2m}\|\boldsymbol{P}\|^2. \tag{72}$$

The corresponding Poisson operator is

$$\mathcal{J} = -\mathcal{P}_\sigma(\nabla \times \boldsymbol{P}) \times . \tag{73}$$

Notice that ρ^{-1} is formally replaced by $\mathcal{P}_\sigma$.

By the same reduction and replacements, as well as appropriate normalization of variables, [21] the incompressible MHD equation is given by a Hamiltonian

$$H = \frac{1}{2}\left(\|\boldsymbol{P}\|^2 + \|\boldsymbol{B}\|^2\right), \tag{74}$$

and a Poisson operator

$$\mathcal{J} = \begin{pmatrix} -\mathcal{P}_\sigma(\nabla \times \boldsymbol{V})\times & \mathcal{P}_\sigma(\nabla \times \circ) \times \boldsymbol{B} \\ \nabla \times [\circ \times \boldsymbol{B}] & 0 \end{pmatrix}. \tag{75}$$

6.3 Casimir elements

In the forgoing examples of Hamiltonian formalisms, the Poisson operators are noncanonical differential operators. Let us find Casimir elements of two typical examples.

[20] More explicitly, φ is determined, for a given $\boldsymbol{u}$, by solving a Poisson equation $\Delta\varphi = \nabla \cdot \boldsymbol{u}$ with a Neumann boundary condition $\boldsymbol{n} \cdot \nabla\varphi = 0$.

[21] We normalize $\rho = 1$, and, by a representative vale of B, $\boldsymbol{B}/B$. And defining a representative velocity V such that $mV^2/2 = B^2/2\mu_0$ (which is called the Alfvén velocity), we normalize $\boldsymbol{V}/V$.

6.3.1 Compressible fluid/plasma model

The Poisson operator (68) of compressible fluid/plasma model (in a plasma, $\boldsymbol{P}$ is the canonical momentum) has two general Casimir elements: [22]

$$C_1 = \int \rho \, d^3x \tag{76}$$

$$C_2 = \frac{1}{2}\int \boldsymbol{P}\cdot(\nabla\times\boldsymbol{P})\, d^3x. \tag{77}$$

Evidently, the constancy of C_1 implies the conservation of total particle number. C_2 is called the *helicity* of the vorticity $\boldsymbol{\omega}$. For C_2 to be a Casimir element, we need an additional boundary condition

$$\boldsymbol{n}\cdot\boldsymbol{\omega} = 0 \quad (\text{on } \partial\Omega).$$

6.3.2 Incompressible MHD model

For the Poisson operator (75) of incompressible MHD, we have two general Casimir elements;

$$C_1 = \frac{1}{2}\langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle \tag{78}$$

$$C_2 = \langle \boldsymbol{V}\cdot\boldsymbol{B}\rangle \tag{79}$$

We call C_1 the *magnetic helicity* and C_2 the *cross helicity*.

Remark 2 *When Ω is multiply connected, the harmonic (vacuum) component included in the magnetic field $\boldsymbol{B}$ must fixed as a constant field, i.e. decomposing $\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{B}_H$ ($\boldsymbol{B}_\Sigma \in L^2_\Sigma(\Omega)$, $\boldsymbol{B}_H \in L^2_H(\Omega)$; see Theorem 4), we have to fix $\boldsymbol{B}_H$ be temporary constant, and define the dynamical state vector to be $u =^t (\rho, m\boldsymbol{V}, \boldsymbol{B}_\Sigma)$. Then, the magnetic helicity must be defined as a functional of $\boldsymbol{B}_\Sigma$. We can find a vector potential $\boldsymbol{A}_H$ of the harmonic field $\boldsymbol{B}_H$ as a member of $L^2_\Sigma(\Omega)$ [11]. And, the vector potential of $\boldsymbol{B}_\Sigma$ can be determined as $\mathcal{S}^{-1}\boldsymbol{B}_\Sigma \in L^2_\Sigma(\Omega)$, where $\mathcal{S} : L^2_\Sigma(\Omega) \to L^2_\Sigma(\Omega)$ is the self-adjoint curl operator [11]. Using these definitions, we define*

$$C_1 = \langle \boldsymbol{A}_H, \boldsymbol{B}_\Sigma\rangle + \frac{1}{2}\langle \mathcal{S}^{-1}\boldsymbol{B}_\Sigma, \boldsymbol{B}_\Sigma\rangle. \tag{80}$$

[22] Here, "general" means that we do not assume any additional conditions on the state vector $u =^t (\rho, \boldsymbol{P})$. If we assume, for example, that u is two-dimensional, then we have additional (in fact, infinite number of Casimir elements.

6.4 Beltrami equilibria

Let us continue with the incompressible MHD model. The Hamiltonian (74) is nothing but the L^2-norm of the phase space $X = L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$. Hence, the equilibrium point of the Hamiltonian is just the *vacuum* ${}^t(\boldsymbol{V}, \boldsymbol{B}) ={}^{(}\, 0, 0)$. However, on the Casimir leaves, we find interesting structures. Combining the Hamiltonian and the Casimir elements (78) and (79), we obtain an energy-Casimir functional

$$H_{\mu_1\mu_2}(u) = H(u) - \mu_1 C_1(u) - \mu_2 C_2(u). \tag{81}$$

The equilibrium point on the Casimir leaves is found by solving $\partial_u H_{\mu_1\mu_2}(u) = 0$, which reads

$$\boldsymbol{V} - \mu_2 \boldsymbol{B} = 0, \tag{82}$$

$$\boldsymbol{B} - \mu_1 \boldsymbol{A} - \mu_2 \boldsymbol{V} = 0. \tag{83}$$

Combining (82) and the curl of (83), we obtain

$$(1 - \mu_2^2)\nabla \times \boldsymbol{B} = \mu_1 \boldsymbol{B}. \tag{84}$$

6.4.1 Beltrami fields

When $\mu_2^2 \neq 1$, (84) reads as the eigenvalue problem of the curl operator.

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}. \tag{85}$$

The solution of (85), i.e. the eigenfunctions of the curl operator is called *Beltrami fields*.

A key of the theory is the formulation of self-adjoint curl operator [11] (for the basic definitions of function spaces, see Appendix B).

Theorem 2 *Let $\Omega \subset \mathbb{R}^3$ be a smoothly bounded domain. We define a curl operator $\mathcal{S}$ in the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$\begin{aligned} \mathcal{S}\boldsymbol{u} &= \nabla \times \boldsymbol{u}, \\ D(\mathcal{S}) &= \{\boldsymbol{u} \in L^2_\Sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}. \end{aligned}$$

Then $\mathcal{S}$ is a self-adjoint operator. The spectrum of $\mathcal{S}$ consists of only point spectra $\sigma_p(\mathcal{S})$, which is a discrete set of real numbers.

If Ω is multiply connected ($m > 0$), a member of $L^2_\Sigma(\Omega)$ has only zero-flux. We have to extend the domain and range of the curl operator to the total space $L^2_\sigma(\Omega)$ of incompressible vector fields to obtain finite-flux Beltrami fields. As an intermediate step, we consider an extended curl operator such that

$$\begin{aligned} \mathcal{T}\boldsymbol{u} &= \nabla \times \boldsymbol{u}, \\ D(\mathcal{T}) &= \{\boldsymbol{u} \in L^2_\Sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\sigma(\Omega)\}. \end{aligned}$$

Lemma 1 *For every $\lambda \in \mathbb{C} \setminus \sigma_p(\mathcal{S})$ and for every $\boldsymbol{f} \in L^2_\sigma(\Omega)$, the equation*

$$(\mathcal{T} - \lambda)\boldsymbol{u} = \boldsymbol{f} \tag{86}$$

has a solution in $L^2_\sigma(\Omega)$.

(proof) First we show the existence of $\mathcal{T}^{-1}$, i.e., for $\boldsymbol{f} \in L^2_\sigma(\Omega)$ we solve

$$\mathcal{T}\boldsymbol{u} = \boldsymbol{f}.$$

Let $\tilde{\boldsymbol{f}}$ be the 0-extension of $\boldsymbol{f}$ in $\mathbb{R}^3$, i.e.

$$\tilde{\boldsymbol{f}}(x) = \begin{cases} \boldsymbol{f}(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

By $\boldsymbol{f} \in L^2_\sigma(\Omega)$, one observes $\nabla \cdot \tilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$. We denote by $(-\Delta)^{-1}$ the vector Newtonian potential. We define

$$\boldsymbol{w}_0 = \nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}] \text{ in } \Omega.$$

We denote by $\mathcal{P}_\Sigma$ the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$, and define

$$\boldsymbol{u}_0 = \mathcal{P}_\Sigma \boldsymbol{w}_0.$$

Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), we observe

$$\nabla \times \boldsymbol{u}_0 = \nabla \times \boldsymbol{w}_0 = \nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}]\}.$$

Since $\nabla \cdot [(-\Delta)^{-1}\tilde{\boldsymbol{f}}] = 0$,

$$\nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{\boldsymbol{f}}]\} = -\Delta[(-\Delta)^{-1}\tilde{\boldsymbol{f}}] = \tilde{\boldsymbol{f}}.$$

We thus have a solution $\mathcal{T}^{-1}\boldsymbol{f} = \nabla \times \boldsymbol{u}_0$.

Next we solve

$$(\mathcal{T} - \lambda)\boldsymbol{u} = \boldsymbol{f},$$

for $\boldsymbol{f} \in L^2_\sigma(\Omega)$. We decompose

$$\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}, \quad [\boldsymbol{g} = \mathcal{P}_\Sigma \boldsymbol{f}, \quad \boldsymbol{h} \in L^2_H(\Omega)].$$

Let $\boldsymbol{u}_0 = \mathcal{T}^{-1}\boldsymbol{h}$ and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_0$. Then (86) reads

$$(T - \lambda)\boldsymbol{w} = \boldsymbol{g} + \lambda \boldsymbol{u}_0 \quad \in L^2_\Sigma(\Omega). \tag{87}$$

For $\lambda \notin \sigma_p(\mathcal{S})$, we may define

$$\boldsymbol{w} = (\mathcal{S} - \lambda)^{-1}(\boldsymbol{g} + \lambda \boldsymbol{u}_0) \quad \in D(\mathcal{S}),$$

which solves (87). In summary, we have a solution of (86)

$$\boldsymbol{u} = \mathcal{T}^{-1}\boldsymbol{h} + (\mathcal{S} - \lambda)^{-1}(\mathcal{P}_\Sigma \boldsymbol{f} + \lambda \mathcal{T}^{-1}\boldsymbol{h}).$$

(Q.E.D.)

Theorem 3 *In $L^2_\sigma(\Omega)$ we define a curl operator $\tilde{\mathcal{S}}$ by*

$$\begin{aligned} \tilde{\mathcal{S}} &= \nabla \times \boldsymbol{u}, \\ D(\tilde{\mathcal{S}}) &= \{\boldsymbol{u} \in L^2_\sigma(\Omega) \ ; \ \nabla \times \boldsymbol{u} \in L^2_\sigma(\Omega)\}. \end{aligned}$$

(i) When $\dim L^2_H(\Omega) = 0$, i.e. if Ω is simply connected, $\tilde{\mathcal{S}} \equiv \mathcal{S}$. Therefore, the spectrum of $\tilde{\mathcal{S}}$ consists of only real point spectra.

(ii) When $\dim L^2_H(\Omega) > 0$, i.e. if Ω is multiply connected, $\tilde{\mathcal{S}}$ is an extension of $\mathcal{S}$. The spectrum of $\tilde{\mathcal{S}}$ consists of only spectra $\sigma_p(\tilde{\mathcal{S}})$, and $\sigma_p(\tilde{\mathcal{S}}) = \mathbb{C}$. In other words, for every $\lambda \in \mathbb{C}$, the uniform Beltrami equation

$$(\tilde{\mathcal{S}} - \lambda)\boldsymbol{u} = 0 \tag{88}$$

has a nontrivial solution.

(proof) The first part is straightforward. We prove the second part. For $\lambda \in \sigma_p(\mathcal{S})$, this has a solution as shown in Theorem 2. We assume $\lambda \notin \sigma_p(\mathcal{S})$. For $\boldsymbol{h} \in L^2_H(\Omega)$,

$$(\mathcal{T} - \lambda)\boldsymbol{v} = \lambda \boldsymbol{h}$$

has a solution (Lemma 1). We easily verify that the function

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{h} \quad \in L^2_\sigma(\Omega) \cap H^1(\Omega)$$

solves (88).

(Q.E.D.)

For the bifurcation of Beltrami equilibria (and tearing mode theory), see [16].

Example 4 (Chandrasekhar-Kendall eigenfunctions) *Let us consider a periodic cylinder domain* Ω*. In the* (r, φ, z) *cylindrical coordinates (*z *is normalized by the longitudinal length of* Ω*), we define*

$$\boldsymbol{u} = \lambda(\nabla\psi \times \nabla z) + \nabla \times (\nabla\psi \times \nabla z), \tag{89}$$

where

$$\lambda = \pm(\mu^2 + k^2)^{1/2}, \tag{90}$$

$$\psi = J_m(\mu r)e^{i(m\varphi - kz)}, \quad m, k \in \mathbb{N}, \tag{91}$$

and J_m *is the Bessel function. We easily find that* $\boldsymbol{u}$ *is an eigenfunction of the curl corresponding to the eigenvalue* λ $(\in \mathbb{R})$*, which we call a* Chandrasekhar-Kendall function *(CK function for short) [1]. The eigenvalue is determined by the boundary condition* $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ *at the surface of* Ω*. This condition becomes trivial when* $k = m = 0$*. For these axisymmetric modes, we invoke the zero-flux condition*

$$\Phi_\Sigma = \int_\Sigma \boldsymbol{n} \cdot \boldsymbol{u} d^2x = 0, \tag{92}$$

where Σ *is a cut of the cylinder, and* $\boldsymbol{n}$ *is the unit normal vector onto* Σ*. Since* $\nabla \cdot \boldsymbol{u} = 0$*, the flux* Φ_Σ *is independent of the place of* Σ*. The totality of CK functions is* complete *to span the function space* $L^2_\sigma(\Omega)$ *of a cylindrical domain* Ω *[12].*

6.4.2 Alfvén waves

When $\mu_2^2 = 1$, (84) has nontrivial solution if $\mu_1 = 0$, and then, $\boldsymbol{B}$ is an arbitrary function and $\boldsymbol{V} = \pm\boldsymbol{V}$. This (infinite-dimensional) set of stationary solutions can be connected to *Alfvén waves* [4]. Let us write this static solution as

$$\boldsymbol{B} = \boldsymbol{B}_0 + \tilde{\boldsymbol{B}} = \boldsymbol{e}_z + \tilde{\boldsymbol{B}}, \tag{93}$$

where $\boldsymbol{e}_z = \nabla z$ is the unit vector parallel to the coordinate z. We interpret that $\boldsymbol{B}_0$ is the homogeneous ambient magnetic field. The coupled flow velocity is, then,

$$\boldsymbol{V} = \boldsymbol{V}_0 + \tilde{\boldsymbol{V}} = \pm(\boldsymbol{e}_z + \tilde{\boldsymbol{B}}). \tag{94}$$

Galilean boost $z \to \zeta = z \mp t$ yields a "propagating wave" with wave fields $\tilde{\boldsymbol{B}}(x, y, \zeta)$ and $\tilde{\boldsymbol{V}}(x, y, \zeta) = \pm\tilde{\boldsymbol{B}}(x, y, \zeta)$ on the ambient magnetic field $\boldsymbol{B}_0 = \boldsymbol{e}_z$, which solves the fully nonlinear MHD equations (66) on the frame (x, y, ζ). In fact, substituting (93) and (94) into (66), we obtain (here we consider an incompressible model)

$$\begin{cases} (\partial_t + \boldsymbol{V}_0 \cdot \nabla)\tilde{\boldsymbol{V}} = -\mathcal{P}_\sigma(\nabla \times \tilde{\boldsymbol{V}}) \times \tilde{\boldsymbol{V}} + \mathcal{P}_\sigma(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \\ (\partial_t + \boldsymbol{V}_0 \cdot \nabla)\boldsymbol{B} = \nabla \times (\tilde{\boldsymbol{V}} \times \boldsymbol{B}). \end{cases} \tag{95}$$

For a boosted quantity $f(\tau, \zeta)$ (with $\tau = t$ and $\zeta = z - V_0 t = z \mp t$), we may write $(\partial_t + \boldsymbol{V}_0 \cdot \nabla) = \partial_\tau$. Therefore, the foregoing static solution appears as a propagating wave on the boosted frame, which solves (66) with transforming $t \to \tau = t$, $z \to \zeta = z \mp t$, and $\boldsymbol{V} \to \tilde{\boldsymbol{V}}$. Since $\tilde{\boldsymbol{B}}$ is arbitrary, perturbations of any shape and any amplitude propagate, with conserving the wave form, at the constant velocity ± 1 (the Alfvén velocity) in the direction of $\boldsymbol{B}_0 = \boldsymbol{e}_z$.

The foregoing is a new-angle derivation of the well-know non-dispersive nonlinear Alfvén waves on a homogeneous ambient magnetic field, by which we notice the fundamental relation between the topological defect of the MHD system and the strikingly robust property of the nonlinear Alfvén waves [15].

7 Conclusion

General theories of physics are composed of two elements: one is space-time and the other is matter. Space-time is formulated by geometry, while matter is represented by energy=Hamiltonian. As for geometry, we have proposed a dichotomy: micro world is canonical=symplectic, while macro world is noncanonical.

We have identified macroscopic scale hierarchy with a foliated phase space of noncanonical Hamiltonian systems. The example of magnetized particles (Sec 4) suggests us to interpret a Casimir element (invariant) as an adiabatic invariant; by coarse-graining (averaging) microscopic angle variable, the corresponding action variable becomes an adiabatic invariant, which can be recognized as a Casimir invariant by noncanonicalizing the corresponding symplectic matrix. We may consider in an opposite way; a Casimir invariant had an adjoint "angle variable". By recovering it, the corresponding kernel of the noncanonical Poisson operator can be removed (Sec. 3.3), and a perturbation of the Hamiltonian with the "angle variable" unfreezes the Casimir invariant.

The canonical micro-world geometry is structured by a symplectic 2-from (field tensor) that is the "vorticity (exterior derivative) of a canonical 1-form. Conversely, we may say that a VORTICITY determines some geometry that dictates dynamics of matter. A fluid vorticity, or a plasma vorticity (which is accompanied by electromagnetic vorticity=magnetic field) determines a "spontaneous" noncanonical geometry that dictates the dynamics of the fluid or plasma itself (Sec. 6.2). By its noncanonicality, interesting structures are self-organized on Casimir leaves (Sec. 6.4; for richer structures of generalized Beltrami fields, see [8] and papers cited there).

APPENDIX

A Lie algebra

The aim of this appendix is to construct intuitive connections between mathematical narratives of space-time and physical understanding of "states" and "geometry". For this purpose, we avoid stating rigorous definitions of mathematical notions —which are available in basic textbooks (for example, see [3])— instead, we try to put them in more accessible terms of ordinary language.

A.1 Process and operation

Let us consider two different "processes" A and B occurring on a system. For example, given a gas in cylinder (a familiar setting of thermodynamics), A is hearting and B is compression. We denote by $\boldsymbol{x} \in X$ an initial "state" (X is a "space" representing a system), and by $A\boldsymbol{x}$ the result of the process A (i.e. A operates from the left). If the process B occurs first, and then A occurs, the state $\boldsymbol{x}$ will be transformed to $AB\boldsymbol{x}$. If the order of processes is reversed, we will obtain $BA\boldsymbol{x}$. When the order of processes (or, the "path" of transformations) changes the results, i.e. $AB \neq BA$, we say that the processes have *hysteresis*. Denoting the *commutation* by

$$[A, B] := AB - BA,$$

non-commutativity of association=product $[A, B] \neq 0$ is the mathematical representation of hysteresis.

The "cycles" (as often discussed in thermodynamics) may be represented as follows: let A^{-1} denote the reversal of the precess A, i.e. $A^{-1}A = AA^{-1} = I$, where the identity I means no change. A cycle composed by two processes A and B is a chain of processes such as $A \to B \to A^{-1} \to B^{-1}$ On a plane, one may represent A by a horizontal movement (A to the right and A^{-1} to the left), and B by a vertical movement (B goes up and B^{-1} goes down), and then, this process draws a anti-clockwise rectangular cycle. By the product representation of associations of processes, this cycle is written as $B^{-1}A^{-1}BA$. If A and B commutes, we may calculate as $B^{-1}A^{-1}BA = B^{-1}A^{-1}AB = I$, i.e. the initial state recovers. However, non-commutativity yields $B^{-1}A^{-1}BA \neq I$. The change caused by a cycle, represented by $(B^{-1}A^{-1}BA - I)\boldsymbol{x}$, is called *circulation*, which is a signature of "vortex" in the space X.

In the foregoing arguments, we have already invoked the algebraic notion of *group*, which is defined as

Definition 1 (group) *A set G (of processes) is called a group, if its all elements satisfy the following conditions:*

1. *a* product AB *is uniquely determined as an element of* G*,*
2. $A(BC) = (AB)C$ *(associative law),*
3. *there is a unique element I (called identity) such that $IA = AI = A$ for every $A \in G$,*
4. *for every $A \in G$, there is a unique element A^{-1} (called inverse) such that $A^{-1}A = AA^{-1} = I$.*

In the foregoing example of thermodynamical processes, each process has an "intensity" parameterized by a real number; for example, heating A may be scaled by a heat Q, and compression B may be scaled by a volume V. Such "continuous" processes constitute a Lie group:

Definition 2 (Lie group) *A continuous set G of processes that are parameterized by real numbers is called a* Lie group*.*

Example 5 (SO(3)) *Let us consider three types of matrices:*

$$\begin{aligned}
A_1(\epsilon) &= \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
A_2(\epsilon) &= \begin{pmatrix} \cos\epsilon & 0 & \sin\epsilon \\ 0 & 1 & 0 \\ -\sin\epsilon & 0 & \cos\epsilon \end{pmatrix}, \\
A_3(\epsilon) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix},
\end{aligned}$$

which represent the processes of rigid-body rotation around the axes z, y, and x, respectively; the real parameter ϵ scales the "intensity" of each process, i.e. the rotation angle. The set $\mathrm{SO}(3) = \{A_1(\epsilon_1), A_2(\epsilon_2), A_3(\epsilon_3)\}$ *is a Lie group.*

[Q] Show that SO(3) satisfies the conditions of Lie group.

[Q] Calculate commutations of the elements of SO(3).

A process $A(\epsilon)$ of is "generated" by some "operation" (in mathematics, we say "operator") a, like a motion is generated by a velocity. To put it in another way, one may identify the causal operation a (which is called the *generator*) of a process $A(\epsilon)$ by evaluating a "differential", i.e.

$$a := \left.\frac{dA(\epsilon)}{d\epsilon}\right|_{\epsilon=0}.$$

The operator a may be regarded as the "tangent vector" (at the "origin" I) of the process $A(\epsilon)$. For each element of a Lie group G, we may identify a tangent vector, and by which we may span the tangent space (at the "origin" I) of the manifold G. The tangent space at the origin I, denoted by $T_I G$, [23] is a vector space endowed with vector calculus and the commutation product $[a, b]$. [24]

Definition 3 (Lie algebra) *The tangent space $\mathfrak{g} := T_I G$ of a Lie group G is a Lie algebra, or a Lie algebra $\mathfrak{g} = \{\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots ; \epsilon_j \in \mathbb{R}\}$ generates a Lie group $G = \{e^{\epsilon_1 a_1}, e^{\epsilon_2 a_2}, \cdots ; \epsilon_j \in \mathbb{R}, a_j \in g\}$.*

Example 6 ($\mathfrak{so}(3)$) *The differential of the processes of* SO(3) *are*

$$\begin{aligned} a_1 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ a_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ a_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

The three-dimensional vector space $\mathfrak{so}(3)$ spanned by $\{a_1, a_2, a_3\}$ is a Lie algebra, which is characterized by a commutation relation

$$[a_1, a_2] = a_3, \quad ((1,2,3)\,\text{cyclic}).$$

[23] Or, we may consider the quotient ring TG/G.

[24] The ring of partial differential operators $\{\alpha_1 \partial_{\epsilon_1} + \alpha_2 \partial_{\epsilon_2} + \cdots ; \alpha_j \in \mathbb{R}\}$ acting on G determines the algebraic structure of the Lie ring $T_I G$. The first order derivatives introduces the sum and scalar multiples of $T_I G$. Second order (and higher-order) derivatives extracts the commutation relations of the products of G, defining the Lie-bracket product $[a, b]$. Notice that the Lie ring's product is not the association such as ab (such product may not be a member of $\mathfrak{g}$), but is the Lie bracket.

A.2 Moving operator and flow / Vector field

Let us consider a state $\boldsymbol{x}$ that is a "point" in a Euclidean space $X = \mathbb{R}^n$. [25] We consider a process that moves the point in the direction of $\boldsymbol{e}^j$ by a magnitude ϵ:

$$A_j(\epsilon) : \boldsymbol{x} \mapsto \boldsymbol{x} + \epsilon \boldsymbol{e}^j.$$

Formally, we may write

$$A_j(\epsilon) = 1 + \epsilon \partial_{x^j}.$$

To make a "group" of such processes, we should put

$$A_j(\epsilon) = e^{\epsilon \partial_{x^j}},$$

and define $G = \{A_1(\epsilon), \cdots, A_n(\epsilon)\}$. The Lie algebra of G is

$$\mathfrak{g} = \{\epsilon_1 \partial_{x^1} + \cdots \epsilon_n \partial_{x^n}\},$$

which is evidently a commutative ring.

Generalizing this "homogeneous movement", we consider processes of arbitrary movements:

Definition 4 (vector field) *A* vector field *is an element of a Lie algebra (called* tangent bundle *of X)*

$$TX = \left\{ v^1(\boldsymbol{x})\partial_{x^1} + \cdots + v^n(\boldsymbol{x})\partial_{x^n};\ v^j(\boldsymbol{x}) \in C^\infty(X) \right\}.$$

An element $v^j(\boldsymbol{x})\partial_{x^j}$ generates a process (infinitesimal flows)

$$T_j(\epsilon) : \boldsymbol{x} \mapsto e^{\epsilon v^j(\boldsymbol{x})\partial_{x^j}} \boldsymbol{x} = \boldsymbol{x} + \epsilon v^j(\boldsymbol{x}) \boldsymbol{e}^j + O(\epsilon^2).$$

[Q] Show that the set TX is "closes" in the sense that every Lie products $[v, w]$ $(v, w \in TX)$ are members of TX.

A.3 Differential forms

A.3.1 covector / cotangent bundle

In the previous subsection, we have formulated a group of moving processes $(T_j(\epsilon))$ and the corresponding Lie algebra of operators $(v^j(\boldsymbol{x})\partial_{x^j})$. When an

[25] One may generalize X to be an n dimensional manifold. Then the following $\boldsymbol{e}^j$ is the unit vector of its local Euclidean coordinate.

infinitesimal process $T_j(\epsilon)$ occurs on a state = *point* $\boldsymbol{x}_s \in X$, an observable (a scalar function) $f(\boldsymbol{x})$ varies as

$$f(\boldsymbol{x}_s) \mapsto f(T_j(\epsilon)\boldsymbol{x}_s) = f(\boldsymbol{x}_s) + v^j(\boldsymbol{x})\partial_{x^j} f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_s}.$$

Hence, the variation (at an arbitrary point) is given by

$$\left.\frac{d}{d\epsilon} f(T_j(\epsilon)\boldsymbol{x})\right|_{\epsilon=0} = v^j(\boldsymbol{x})\partial_{x^j} f(\boldsymbol{x}).$$

Generalized for an arbitrary movement $T_{\boldsymbol{v}}(\epsilon)$ generated by a flow (vector) $\boldsymbol{v} = v^1(\boldsymbol{x})\partial_{x^1} + \cdots + v^n(\boldsymbol{x})\partial_{x^n}$, we may write

$$\left.\frac{d}{d\epsilon} f(T_{\boldsymbol{v}}(\epsilon)\boldsymbol{x})\right|_{\epsilon=0} = \sum_j v^j(\boldsymbol{x})\partial_{x^j} f(\boldsymbol{x}).$$

One may see this as the application of a differential operator $(\boldsymbol{v} \cdot \nabla)$ on a scalar function f.

Another interpretation is, denoting

$$\sum_j v^j(\boldsymbol{x})\partial_{x^j} f(\boldsymbol{x}) = \boldsymbol{v} \cdot (\nabla f),$$

an association of a vector $\boldsymbol{v}$ and a *covector* ∇f (vectors and covectors are *dual*; their associations yield real numbers).

The covector ∇f is a member of the *cotangent bundle* T^*X, the dual space of TX, whose member is written as

$$\boldsymbol{\omega} = \omega_1 dx^1 + \omega_2 dx^2 + \cdots + \omega_n dx^n.$$

The duality of TX and T^*X implies

$$\partial_{x^j} dx^k = \delta_{jk}. \tag{96}$$

The association of TX and T^*X is called an *interior product.*

A.3.2 differential forms, exterior products, exterior derivatives

In the above calculation of the variation of a scaler function $f(\boldsymbol{x})$ (we will denote the space of scalar functions as $\mathrm{Fun}(X)$), we considered a special covector ∇f, which may be regarded as an *image* of a linear map

$$d : \mathrm{Fun(X)} \to T^*X,$$

which is explicitly defined by

$$df(\boldsymbol{x}) = \sum_j \partial_{x^j} f(\boldsymbol{x})\, dx^j.$$

We call this differential operator d an *exterior derivative.*

A general member of T^*X is not necessarily an exterior derivative of a scalar function. We may regard them also physical quantities; we call them (differential) "1-forms". We may further generalize the notion of physical quantities by introducing *exterior products* of 1-forms (or T^*X): For a pair of 1-forms $\alpha = \sum \alpha_j dx^j,\ \beta = \sum \beta_j dx^j$, we define their exterior product by

$$\alpha \wedge \beta = \sum_{j,k=1}^{n} \alpha_j \beta_k\ dx^j \wedge dx^k. \tag{97}$$

Here, the *wedge product* $\wedge$ is defined by an antisymmetric relation

$$dx^j \wedge dx^k = -dx^k \wedge dx^j. \tag{98}$$

Evidently, $dx^j \wedge dx^j = 0$. By (98), we may rewrite (97) as

$$\alpha \wedge \beta = \sum_{j<k} (\alpha_j \beta_k - \alpha_k \beta_j) dx^j \wedge dx^k.$$

The exterior product $\alpha \wedge \beta$ (which we call a 2-form) is a member of the $\wedge^2 T^*X$, which is a linear space of dimension $\binom{n}{2}$ having a basis $\{dx^j \wedge dx^k; 1 \leq j < k \leq n\}$.

Continuing the exterior products, we may define p-forms. The linear space $\wedge^p T^*X$ of p-forms is of dimension $\binom{n}{p}$ (hence, $p \leq n$), and has a basis

$$\{\mathrm{sgn}(\sigma_j) dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_p};\ 1 \leq j_1 < \cdots < j_p \leq n\}, \tag{99}$$

$\mathrm{sgn}(\sigma_j)$ is the sign of transpose σ_j.

We are ready to generalize the exterior derivative d to be differential operators mapping a p-form to a $(p+1)$-form. For an r-form $(r \leq n-1)$

$$\omega = \sum_{j_1<j_2<\cdots<j_r} \omega_{j_1\cdots j_r} dx^{j_1} \wedge \cdots \wedge dx^{j_r},$$

we define

$$d\omega = \sum_{j_1<j_2<\cdots<j_r} (d\omega_{j_1\cdots j_r}) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_r},$$

which is an $(r+1)$-form. Or, we may put

$$d\omega = \sum_{\ell} dx^{\ell} \wedge (\partial_{x^\ell} \omega).$$

[Q] Consider $n = 2, 3$ and 4, and find what are the d in each dimension.

A.4 Lie derivatives

For a scalar-function observable $f(\boldsymbol{x})$ (0-form $\in$ X), a state is represented by a point $\boldsymbol{x}_s \in X$, and the physical quantity evaluates as

$$f(\boldsymbol{x}_s) := f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_s} = \langle f, \delta(\boldsymbol{x} - \boldsymbol{x}_s)\rangle. \tag{100}$$

As we have seen, a flow $T_{\boldsymbol{v}}(\epsilon)$ moves $\boldsymbol{x}_s$ and causes a variation of $f(\boldsymbol{x}_s)$ as (evaluating for general $\boldsymbol{x}$)

$$\left.\frac{d}{d\epsilon} f(T_{\boldsymbol{v}}(\epsilon)\boldsymbol{x})\right|_{\epsilon=0} = \sum_j v^j(\boldsymbol{x})\partial_{x^j} f(\boldsymbol{x}).$$

We have introduced the notion of "interior product" between the vector $v \in TX$ and covector $df \in T^*X$, which we denote

$$i_v df.$$

The left-hand side is called a *Lie derivative* associated with the flow v, which we denote by $L_v f$, i.e. for a scalar function (0-form), we write the variation caused by a flow v as

$$L_v f = i_v f.$$

In the preceding subsection, we have generalized the notion of physical quantity to p-forms. As a state corresponding to a 0-form physical quantity is a "point", a 0-dimensional geometric object, a state corresponding to a p-form physical quantity (ω) must be a p-dimensional geometric object Ω, and its value is evaluated as, generalizing (100)

$$\omega(\boldsymbol{x})|_\Omega = \int_\Omega \omega. \tag{101}$$

The process $T_v(\epsilon)$ now moves Ω. The resultant variation is

$$\int_\Omega \omega \mapsto \int_{T_v(\epsilon)\Omega} \omega,$$

and its derivative with respect to ϵ defines the Lie derivative $\int_\Omega L_u\omega$.

Let us calculate $L_u\omega$ explicitly. We denote $T_v(\epsilon)\Omega = \Omega_\epsilon$. We also denote by X_ϵ the domain over which $T_v(\epsilon)\Omega$ sweeps as ϵ increases, which as a $p+1$-dimension manifold and its boundary is

$$\partial X_\epsilon = \Omega_\epsilon \ \cup \ -\Omega \ \cup \ \epsilon\partial\Omega \times u.$$

Using these notations, we may calculate

$$\begin{aligned}
\frac{d}{d\epsilon}\int_{T_v(\epsilon)\Omega}\omega &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Omega_\epsilon-\Omega}\omega \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{\partial X_\epsilon}\omega - \int_{\epsilon\partial\Omega\times u}\omega\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{X_\epsilon}d\omega + \epsilon\int_{\partial\Omega}i_u\omega\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\epsilon\int_{u\times\Omega}d\omega + \epsilon\int_{\Omega}d(i_u\omega)\right] \\
&= \int_{\Omega}(i_u d\omega + d i_u\omega).
\end{aligned}$$

Hence, we may write

$$L_u\omega = (i_u d + d i_u)\omega, \tag{102}$$

where i_v is the *interior product* with the vector v based on the duality (96), which maps a p-form to a $(p-1)$-form (for a 0-form f, we define $i_u f = 0$).

B Orthogonal decomposition of $L^2(\Omega)$

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider cuts of the domain Ω. Let $\Sigma_1 \cdots \Sigma_m$ $(m \geq 0)$ be cuts such that $\Sigma_i \cap \Sigma_j = \emptyset$ $(i \neq j)$, and such that $\Omega \setminus (\cup_{i=1}^m \Sigma_i)$ becomes a simply connected domain. The number m of such cuts is the first Betti number of Ω.

When $m > 0$, we define the flux through each cut by

$$\Phi_{\Sigma_i} = \int_{\Sigma_i} \boldsymbol{n}\cdot\boldsymbol{u} d^2x, \quad (i = 1, 2, \cdots, m),$$

where $\boldsymbol{n}$ is the unit normal vector on Σ_i with an appropriate orientation. By Gauss's formula Φ_{Σ_i} is independent of the place of the cut Σ_i, if $\nabla\cdot\boldsymbol{u} = 0$ in Ω and $\boldsymbol{n}\cdot\boldsymbol{u} = 0$ on $\partial\Omega$.

We define the following subspaces of $L^2(\Omega)$;

$$\begin{aligned}
L^2_\Sigma(\Omega) &= \{\boldsymbol{w};\ \nabla\cdot\boldsymbol{w} \text{ in } \Omega, \boldsymbol{n}\cdot\boldsymbol{w} = 0 \text{ on } \partial\Omega, \forall\Phi_{\Sigma_i} = 0\}, \\
L^2_H(\Omega) &= \{\boldsymbol{h};\ \nabla\cdot\boldsymbol{h} = 0, \nabla\times\boldsymbol{h} = 0 \text{ in } \Omega, \boldsymbol{n}\cdot\boldsymbol{h} = 0 \text{ on } \partial\Omega\}, \\
L^2_G(\Omega) &= \{\nabla\phi;\ \Delta\phi = 0 \text{ in } \Omega\}, \\
L^2_F(\Omega) &= \{\nabla\phi;\ \phi = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

Here each subspaces are defined by taking the closure in $L^2(\Omega)$ of the set of smooth functions which satisfy the specified relations.

Theorem 4 *We have an orthogonal decomposition [2, 9]*

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_G(\Omega) \oplus L^2_F(\Omega).$$

Evidently,

$$\begin{aligned} \mathrm{Ker}(\mathrm{div}) &= L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_G(\Omega), \\ \mathrm{Ker}(\mathrm{curl}) &= L^2_H(\Omega) \oplus L^2_G(\Omega) \oplus L^2_F(\Omega). \end{aligned}$$

The space of solenoidal vector fields with vanishing normal component on the boundary is given by

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega).$$

This relation is called the Hodge-Kodaira decomposition. The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\mathrm{dim} L^2_H(\Omega) = m$ (the first Betti number of Ω). [26] [27] When Ω is simply connected, then $m = 0$ and $L^2_H(\Omega) = \emptyset$. We have the following expression

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{w};\ \boldsymbol{w} \in H^1(\Omega),\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega, \boldsymbol{n} \times \boldsymbol{w} = 0 \text{ on } \partial\Omega\}.$$

This implies that a function in $L^2_\Sigma(\Omega)$ can be expressed by the curl of a vector potential with the boundary condition $\boldsymbol{n} \times \boldsymbol{w} = 0$. We note that a member of $L^2_\sigma(\Omega)$ may not allow such an expression.

We also note

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus \{\nabla\phi;\ \phi \in H^1(\Omega)\},$$

which implies that $L^2_\sigma(\Omega)$ is the orthogonal complement of the space of *gradient fields* [10]. This relation is called the Weyl decomposition.

[26] Basic relations between harmonic forms and the cohomology classes were studied by KODAIRA, K. (1948) Ann. of Math. **50**, 587. For the theory of differential forms on manifolds with boundaries, see DUFF, G. F. D. (1952) Ann. of Math. **56**, 115; DUFF, G. F. D. & SPENCER, D. C. (1952) Ann. of Math. **56**, 129; CONNOR P. E. (1954) Proc. Nat. Acad. Sci. **40**, 1151. See also Theorem 7.7.7 of MORREY, C. B. (1966) *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, Heidelberg, New York.

[27] In electromagnetism, $L^2_H(\Omega)$ corresponds to the space of vacuum magnetic field with vanishing normal component at the wall; see WERNER, P. (1983) J. Math. Anal. Appl. **92**, 1; KOTIUGA, P. R. (1987) J. Appl. Phys. **61**, 3916.

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