Geometrical Theory of Vortex
— Link in Minkowski Space-Time —

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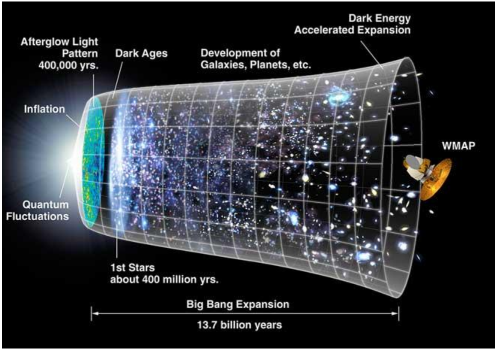
5. We delineate the topological meaning of the relativistic helicity by analyzing the linking number of "vortex filaments" (*pure states* of non-commutative Banach algebra).
6. The non-conservation of the conventional helicity is because vortex filaments are no longer pure states in relativistic dynamics.

Background I

vortex — common *structure* in the Universe



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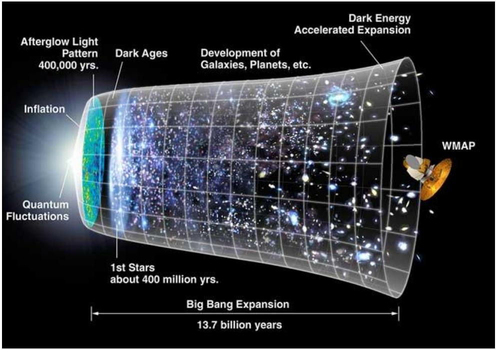

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How was the first vortex created?

Background II (Kelvin's circulation theorem)

- For a *co-moving* loop $L(t)$, the rate of change of the circulation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{L(t)} \mathbf{P}\cdot\boldsymbol{\ell}\mathrm{d}\xi = \oint_{L(t)} [\partial_t \mathbf{P} - \mathbf{v}\times(\nabla\times\mathbf{P})]\cdot\boldsymbol{\ell}\mathrm{d}\xi.$$

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- In a *barotropic fluid*,

$$\partial_t\mathbf{P}-\mathbf{v}\times(\nabla\times\mathbf{P})=-\nabla(h+mv^2/2).$$

Thus, we obtain *Kelvin's circulation theorem*:

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- Neglecting the electron mass (MHD model), we obtain the circulation theorem for the magnetic field, i.e. *Alfvén's theorem*.

Background III (baroclinic effect)

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3. Hence, a momentum (1-form) generated by a scalar (0-form) is naturally "exact", posing a challenge of generating a vortex (2-form).
4. The space-time distortion by the relativistic effect, however, brings about a *relativistic baroclinic effect*, breaking the exactness of the thermal force:

$$(\partial_t + L_{\mathbf{v}}) P = \gamma^{-1} \mathrm{d}\varepsilon.$$

   This mechanism can create a *seed magnetic field (EM vorticity)* in a cosmological plasma; [Mahajan-Yoshida, PRL **105** (2010), 095005]

Background IV (helicity)

conventional definition of helicity

- The conventional helicity of $\mathbf{b} = \nabla \times \mathbf{a}$ is (on a fixed $\Omega \subseteq \mathbb{R}^3$)

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- In a barotropic fluid, $\mathbf{P}$ obeys

$$\partial_t \mathbf{P} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla \varepsilon, \tag{2}$$

where $\varepsilon = h + mv^2/2 + \phi$ ($h(\rho)$: enthalpy, ϕ: E potential).

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where $\varepsilon = h + mv^2/2 + \phi$ ($h(\rho)$: enthalpy, ϕ: E potential).
- Under a boundary condition $\mathbf{n} \cdot \mathbf{b} = 0$, C is conserved.

Background V (linking number)

conventional definition in $\mathbb{R}^3$

- To delineate the topological meaning of C in the simplest form, consider a pair of *vortex filaments*:

$$\mathbf{b}\mathrm{d}^3x = \boldsymbol{\ell}_1\mathrm{d}\xi_1 + \boldsymbol{\ell}_2\mathrm{d}\xi_2,$$

  where $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ are δ-measures on loops Γ_1 and Γ_2.

- By (generalized) Stokes' formula,

$$C = \int_{\mathbb{R}^3} \mathbf{a}\cdot\mathbf{b}\,\mathrm{d}^3x = \oint_{\Gamma_1} \mathbf{a}\cdot\boldsymbol{\ell}_1\mathrm{d}\xi_1 + \oint_{\Gamma_2} \mathbf{a}\cdot\boldsymbol{\ell}_2\mathrm{d}\xi_2 = 2\mathcal{L}(\Gamma_1,\Gamma_2). \quad (3)$$

- By Biot-Savart integral, we may write

$$C = 2\times\frac{1}{4\pi}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{(\mathbf{x}_1-\mathbf{x}_2)\cdot\boldsymbol{\ell}_1\mathrm{d}\xi_1\times\boldsymbol{\ell}_1\mathrm{d}\xi_2}{|\mathbf{x}_1-\mathbf{x}_2|^3}. \quad (4)$$

Basic definitions about Minkowski space-time (1)

- We denote the Minkowski space-time by $M \cong \mathbb{R}^4$; on a reference frame, we write

$$x^\mu = (ct, x, y, z), \quad x_\mu = (ct, -x, -y, -z).$$

- The space-time gradients are denoted by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\nabla \right).$$

- The relativistic 4-velocity is defined by the proper-time derivative:

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = (\gamma, \gamma \mathbf{v}/c), \quad U_\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}s} = (\gamma, -\gamma \mathbf{v}/\mathbf{c}),$$

where $\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Basic definitions of Minkowski space-time (2)

- The fluid 4-momentum is a 1-form $P = P_\mu \mathrm{d}x^\mu \in T^*M$ with $P_\mu = (h/c)\, U_\mu$ (h is the molar enthalpy, h/c^2 is the effective mass density).
- For a charged fluid (plasma), the canonical 4-momentum is $\mathcal{P} = P + qA$, where q is the charge.
- The 4-velocity U^μ is a vector field $U = U^\mu \partial_\mu \in TM$, which generates a diffeomorphism $\mathcal{T}_U(s)$ by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}_U(s) = U, \tag{5}$$

- The "t-plane cross-section" is, for a fixed parameter $t \in \mathbb{R}$,

$$\Xi(t) = \{(x^0, x^1, x^2, x^3);\, x^0 = ct,\, (x^1, x^2, x^3) \in X\}. \tag{6}$$

- The "proper time s-plane cross-section" is

$$\tilde{\Xi}(s) = \mathcal{T}_U(s)\Xi(0). \tag{7}$$

Relativistic Fluid/Plasma Equation

- In terms of the canonical momentum $\mathcal{P} = P + qA$, the matter-EM field tensor is a 2-form

$$\mathcal{M} = \mathrm{d}\mathcal{P} = \partial_\mu \mathcal{P}_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

- The equation of motion reads, assuming a barotropic relation $T\mathrm{d}S = \mathrm{d}\theta$,

$$i_U \mathcal{M} = -c^{-1}\mathrm{d}\theta. \tag{8}$$

- Or, invoking the Lie derivative,

$$L_U \mathcal{P} = c^{-1}\mathrm{d}(h + q\varrho - \theta). \tag{9}$$

Notation (EM analogy)

- The canonical momentum (or dressed EM potential) is

$$\mathcal{P}^\mu = (\mathcal{P}^0, \boldsymbol{\mathcal{P}}) \equiv \mathcal{A}^\mu = (\mathcal{A}^0, \boldsymbol{\mathcal{A}}). \tag{10}$$

- EM vectors:

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= -\nabla \mathcal{A}^0 - (1/c)\partial_t \boldsymbol{\mathcal{A}}, \\ \boldsymbol{\mathcal{B}} &= \nabla \times \boldsymbol{\mathcal{A}}. \end{aligned}$$

- Field tensor:

$$\mathcal{M}^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & -\mathcal{B}_3 & \mathcal{B}_2 \\ \mathcal{E}_2 & \mathcal{B}_3 & 0 & -\mathcal{B}_1 \\ \mathcal{E}_3 & -\mathcal{B}_2 & \mathcal{B}_1 & 0 \end{pmatrix}. \tag{11}$$

Semi-Relativistic Helicity (Nöther charge)

- The helicity $C = \int_X \mathbf{a} \cdot \mathbf{b} \, \mathrm{d}^3 x$ is naturally generalized as

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- We find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} C = c \int_X \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{B}} \, \mathrm{d}^3 x &= 2 \int_X \gamma^{-1} \boldsymbol{\mathcal{B}} \cdot \nabla \theta \, \mathrm{d}^3 x \\ &= -2 \int_X \theta \boldsymbol{\mathcal{B}} \cdot \nabla \gamma^{-1} \, \mathrm{d}^3 x, \end{aligned} \tag{13}$$

showing that the relativistic factor γ can break the constancy of the helicity.

Relativistic Helicity

a new constant in a relativistic plasma

We define a *generalized helicity* $\mathfrak{C}$ in the Minkowski space-time by the integral of the 3-form $\mathcal{K} = \mathcal{P} \wedge \mathrm{d}\mathcal{P}$ over a co-moving 3D spatial volume $V(s) = \mathcal{T}_U(s) V_0$ ($V_0 \in \Xi(0)$):

$$\mathfrak{C}(s) = \int_{V(s)} \mathcal{P} \wedge \mathrm{d}\mathcal{P}. \tag{14}$$

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Theorem

The helicity $\mathfrak{C}(s)$ is a constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{C}(s) = 0.$$

What does the helicity conservation constrain?

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5. The link of surfaces yields a topological constraint on loops (vortex-filaments) that are the *s*-plane (or *t*-plane) cross sections of the vorticity surfaces.

Pure state of differential forms

A generalization for non-commutative Banach algebra

Definition (pure sate)

Let M be a smooth manifold of dimension n, and $\Omega \subset M$ be a p-dimensional connected null-boundary submanifold of class C^1. Each Ω is equivalent to a *pure-sate functional* η_Ω on $\wedge^p T^* M$:

$$\eta_\Omega(\omega) = \int_\Omega \omega = \int_M \mathfrak{J}(\Omega) \wedge \omega,$$

where $\mathfrak{J}(\Omega) = \wedge^{n-p} \delta(x^\mu - \xi^\mu) \mathrm{d}x^\mu$ is a δ-measure supported on Ω. We call $\mathfrak{J}(\Omega)$ a *pure state* $(n-p)$-form, which is a member of the Hodge-dual space of $\wedge^p T^* M$.

Pure state vorticity and **B**-filament

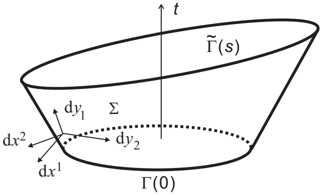
- A pure-state functional $\eta_\Sigma(\omega) = \int_\Sigma \omega$ of vorticity 2-forms ω is

$$\mathfrak{J}(\Sigma) = \delta_\Sigma * \mathrm{d}y_1 \wedge \mathrm{d}y_2 = \delta_\Sigma \mathfrak{m}, \quad \mathfrak{m} = \frac{1}{2}\mathfrak{m}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \qquad (15)$$

  where δ_Σ is the 2D δ-function supported on a surface Σ,
- On the s-plane cross-section $\tilde{\Gamma}(s) = \tilde{\Xi}(s) \cap \Sigma$, we obtain a pure-state "relativistic **B**-filament", which is a singular 3-form such that

$$\mathfrak{J}_b(\tilde{\Gamma}(s)) = \tilde{\rho}_b(s)\mathfrak{J}(\Sigma) = -\delta_{\tilde{\Xi}(s)}\mathcal{U} \wedge \mathfrak{J}(\Sigma), \qquad (16)$$

  where $\mathcal{U} = U_\mu \mathrm{d}x^\mu$.



Helicity and circulation

1. Consider a pair of disjoint loops $\tilde{\Gamma}_1(s)$ and $\tilde{\Gamma}_2(s)$, and their orbits $\Sigma_1 = \bigcup \tilde{\Gamma}_1(s)$ and $\Sigma_2 = \bigcup \tilde{\Gamma}_2(s)$.
2. On each Σ_ℓ, we give a pure-state vorticity $\mathcal{M}_\ell = \mathfrak{J}(\Sigma_\ell)$.
3. Then, on each $\tilde{\Gamma}_\ell(s)$, we obtain a pure-sate B-filament $\mathfrak{J}_b(\tilde{\Gamma}(s))$.
4. Denoting $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ and $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 = \mathcal{F}\mathcal{M}_1 + \mathcal{F}\mathcal{M}_2$, the relativistic helicity of the twin vorticity evaluates as

$$\mathfrak{C}(s) = \int_{V(s)} \mathcal{P} \wedge \mathcal{M} = \int_{\tilde{\Gamma}_1(s)} \mathcal{P}_2 + \int_{\tilde{\Gamma}_2(s)} \mathcal{P}_1. \tag{17}$$

The LW integral operator

Lemma (LW integral)

We denote $\delta = *\mathrm{d}*$, *and* $\Box = \delta\mathrm{d} + \mathrm{d}\delta$ *(d'Alembertian). We invert* $\Box$ *by the Liénard-Wiechert integral operator, which we denote by* $\Box^{-1}$. *We can define*

$$\mathcal{P} = \mathcal{F}\mathcal{M} = \Box^{-1}\delta\mathcal{M}. \tag{18}$$

Link in Minkowski space-time

Theorem (link in Minkowski space-time)

Let $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ be a twin vortex generated by a pair ($\ell = 1, 2$) of pure-state **B***-filaments $\mathfrak{J}_b(\tilde{\Gamma}_1)$ and $\mathfrak{J}_b(\tilde{\Gamma}_2)$.*

1. *The relativistic* **B***-filaments $\mathfrak{J}_b(\tilde{\Gamma}_\ell(s))$ continue to be pure states.*
2. *The relativistic helicity*

$$\mathfrak{C}(s) = \int_{\tilde{\Gamma}_1(s)} \mathcal{F}\mathcal{M}_2 + \int_{\tilde{\Gamma}_2(s)} \mathcal{F}\mathcal{M}_1. \tag{19}$$

   is a constant of motion.

3. *The constant $\mathfrak{C}(s)/2$ is the linking number $\mathcal{L}(\tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s))$, which may be represented as (generalizing Gauss' integral)*

$$\mathcal{L}(\tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s)) = \int \mathcal{F}\mathcal{M}_2 \wedge \mathfrak{J}_b(\tilde{\Gamma}_1(s)) = \int \mathcal{F}\mathcal{M}_1 \wedge \mathfrak{J}_b(\tilde{\Gamma}_2(s)). \tag{20}$$

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5. ZY, Y. Kawazura and T. Yokoyama, *Relativistic helicity and link in Minkowski space-time*; arXiv:1308.2455