Regularization of the Alfvén singularity by the Hall effect

Jun-ya Shiraishi,* Shuichi Ohsaki,† and Zensho Yoshida‡

Graduate School of Frontier Sciences, The University of Tokyo,
5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8561, Japan

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Abstract

An ideal magnetohydrodynamics (MHD) equilibrium is described by a system of rather complicated singular differential equations when a flow is included. The so-called Alfvén singularity occurs at the place where the Doppler-shifted Alfvén velocity vanishes. It is due to the vanishing of the highest order derivative in the differential equation. The Hall effect, working as a singular perturbation to the ideal MHD system, yields a new branch of regular solutions that can smoothly connect two regions separated by the Alfvén singularity. The thickness of the transition layer is of the order of the ion skin depth, the intrinsic length scale brought about by the singular perturbation. The regularization mechanism of the nonlinear Hall effect is not as simple as that of the diffusion effect producing an “entropy (viscosity) solution” in a viscous fluid. The Hall effect removes the restriction binding the magnetic and flow characteristics, and creates the new branch of regularized solutions. One dimensional analysis loses sight of this new branch of equilibria.

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*Electronic address: shira@ppl.k.u-tokyo.ac.jp
†Present address: Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712; Electronic address: ohsaki@mail.utexas.edu
‡Electronic address: yoshida@k.u-tokyo.ac.jp
I. INTRODUCTION

In any theory of dynamical systems, one starts from analyzing an equilibrium state. It is generally a nonlinear problem, and, quite often, it is rather difficult to find how a unique state is selected from a variety of possible solutions. Translating this question into mathematical terms, one has to supply an appropriate set of integral data to select a particular solution from the general solutions of the determining differential equations.

An ideal magnetohydrodynamic (MHD) equilibrium is governed by a system of nonlinear partial differential equations (PDEs) consisting of both elliptic and hyperbolic parts [1]. This system has some fatal mathematical difficulties, and only a very limited analysis has been completed. This system of PDEs is not well-posed as a boundary value problem – the integration of the hyperbolic parts demands so-called “Cauchy data” [2]. The integrability of the Cauchy characteristics is warranted only when the number of independent variables is less than or equal to two (the space dimension is less than or equal to two). Only in two-dimensional systems, and moreover, assuming no flow (static equilibrium), we may integrate the hyperbolic parts to obtain a single elliptic PDE governing the flux function $\psi$, viz., the Grad-Shafranov equation [2]; the profiles of pressure and force-free current must be supplied as the Cauchy data to write down the Grad-Shafranov equation. Then a boundary data of $\psi$ suffices to determine the equilibrium.

Under the existence of flows, the system of MHD equilibrium equations becomes considerably complicated (for example, see Chap. 2 of Ref. [3]), and no general theory has been established for two-dimensional systems (three-dimensional ones are far beyond the scope of mathematical analysis, because of the possible “chaos” of the Cauchy characteristics). The fundamental difficulty is the possibility of a “singularity” caused by the flow. Integrating the hyperbolic parts with supplying Cauchy data, the system of equations may be cast into a single PDE governing $\psi$. This “generalized Grad-Shafranov equation” is no longer an elliptic PDE; it becomes alternately elliptic or hyperbolic depending on the poloidal flow speed [4] (the term “poloidal” means the projection onto the two-dimensional surface); see Sec. II. Moreover, the equation is singular when the poloidal flow speed coincides with the local poloidal Alfvén speed, which is called the “Alfvén singularity” [5]. When the singularity occurs, we cannot construct a global solution; we do not have a physical principle to connect different regions separated by the singularity.
In this paper, we overcome the problem of the singularity by introducing a “singular perturbation” representing some small-scale effect. In a collision-less plasma, the “Hall effect” is the primary candidate of the singular perturbation. We will show that the Hall effect yields a new branch of regular solutions that can smoothly connect two regions separated by the Alfvén singularity. The thickness of the transition layer is of the order of the ion skin depth $\delta_i$, the intrinsic length scale brought about by the singular perturbation. This scenario of removing the singularity is common to the standard singular perturbation theories such as the well known “entropy (or viscosity) solution” in a slightly viscous fluid. However, the regularization mechanism of the Hall effect is not so simple, because it appears as a nonlinear term.

Before the analysis of general two-dimensional problems, we shortly review related theories. Some “self-similar solutions” of the ideal MHD equilibria have been constructed by reducing the PDEs into one-dimensional (ordinary) differential equations [5–7]. The self-similar analysis evades the difficulty of imposing boundary conditions by exploiting the concept of self-similarity (the scale invariance of the solutions), which is considered as a way to establish the solutions valid “far away” from the boundaries [6]. Thus, the self-similar approach cannot capture the global structure of the solutions. In this theory, the standing shocks (the above-mentioned elliptic-hyperbolic transitions) are of the main interest. Avoiding the Alfvén singularity, two-dimensional problems of the ideal MHD equilibria with incompressible flows have been studied analytically [8, 9]. Two-dimensional numerical solutions with compressible flows have been obtained, which are limited in the elliptic regime [10, 11].

The Hall-MHD equilibrium equations have a particular three-dimensional solution, the “double Beltrami field” that is represented by a linear combination of two Beltrami fields (eigenfunctions of the curl operator) [12]. This solution assumes homogeneous Cauchy data that trivialize the integration of the hyperbolic parts even for chaotic characteristics in three-dimensional systems. The eigenvalue of the curl operator is the reciprocal length scale of the corresponding Beltrami field. The double Beltrami solution shows that the Hall effect yields a coupling of different scales [13]. One component of the double Beltrami field may have an arbitrary length scale, and solves the ideal MHD equilibrium equations. The other component is singular in the limit of $\delta_i \to 0$.

More general (in terms of inhomogeneous Cauchy data) two-dimensional Hall-MHD equi-
librium equations have been formulated. For compressible flows, these equations become regular PDEs that have no singularity (except for the region where the density vanishes) [3], and become elliptic when the poloidal flow speed is less than the sound speed [14]. If we assume incompressible flows, the equilibrium equations become two regular elliptic PDEs [13]. It is, thus, obvious that the Hall term may regularize the Alfvén singularity of the ideal MHD system. However, as we will show in this paper, the relation between the regularized equations and the original singular equations is not so simple – the regularized solutions appear as a new branch of equilibria, which is inhibited in the ideal limit. In this paper, we will study the structure of the regularized solutions, and will show how it connects the different regions separated by the singularity.

This paper is organized as follows. In Sec. II, we review the analysis of Cauchy characteristics of the ideal MHD equations, and clarify the physical meaning of the Alfvén singularity of the flowing equilibrium equation. In Sec. III, we introduce the Hall-MHD equilibrium equations. Assuming two-dimensional geometry and incompressibility, we integrate the hyperbolic parts and derive a system of elliptic equations. Through this derivation, we show that there are two different branches of equilibria; one contains the Alfvén singularity, while the other is regular. The latter new branch is characterized by oblique magnetic and flow characteristics. In Sec. IV, we give numerical solutions, and analyze how the solutions are connected across the singularity.

II. ANALYSIS OF THE CAUCHY CHARACTERISTICS

A. Characteristics of the ideal MHD system

We start with analyzing the Cauchy characteristics of the ideal MHD equations [15]

\[
\begin{align*}
\partial_t n + \nabla \cdot (n \mathbf{V}) &= 0, \\
\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{V} + \nabla p &= (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{V}) &= 0, \\
\partial_t p + \mathbf{V} \cdot \nabla p + nC_s^2 \nabla \cdot \mathbf{V} &= 0,
\end{align*}
\]

where \( n, \mathbf{B}, \mathbf{V}, p \) and \( C_s \) are the number density, the magnetic field, the flow velocity, the total pressure and the sound speed, respectively. We have normalized the coordinates \( x \)
by the system size $L_0$, $n$ by the representative number density $n_0$, $B$ by the representative magnetic field $B_0$, $V$ and $C_s$ by the Alfvén speed $V_A = B_0/\sqrt{\mu_0 M n_0}$ ($\mu_0$ is the vacuum permeability and $M$ is the ion mass), $p$ by $B_0^2/\mu_0$ and the time $t$ by $L_0/V_A$.

For this system of first-order quasi-linear symmetric hyperbolic PDEs, the Cauchy characteristics are given by the eikonals [16]. The characteristic manifolds are determined by $\varphi(x, y, z, t) = \text{constant}$ in the $x, y, z, t$ space-time, where $\varphi$ is the eikonal function (the phase of wave propagation). Denoting $\omega = \partial_t \varphi$, $k = \nabla \varphi$, and $\varpi = \omega + k \cdot V$, the characteristic equation reads [1]

$$\omega \varpi \left[ \varpi^2 - \left( k \cdot B \right)^2 / n \right] \left[ \varpi^4 - k^2 \left( C_s^2 + B^2 / n \right) \varpi^2 + k^2 C_s^2 \left( k \cdot B \right)^2 / n \right] = 0. \quad (5)$$

The first root $\omega = 0$ is removed when $\nabla \cdot B = 0$ is taken as a constraint [1, 15]. The second root $\varpi = 0$ corresponds to the “entropy wave.” The last factor of (5) yields the Doppler-shifted fast and slow waves. The remaining roots given by $\varpi^2 - \left( k \cdot B \right)^2 / n = 0$ [equivalent to $\omega + k \cdot (V \pm B/\sqrt{n}) = 0$] describe the Doppler-shifted Alfvén waves.

**B. Characteristics of the flowing ideal MHD equilibrium equations**

The equilibrium equations are obtained by putting $\partial_t = 0$ in (1)-(4), and the corresponding characteristic equation is

$$[k \cdot V] \left[ \left( k \cdot V \right)^2 - \left( k \cdot B \right)^2 / n \right] \times$$

$$\left[ \left( k \cdot V \right)^4 - k^2 \left( C_s^2 + B^2 / n \right) \left( k \cdot V \right)^2 + k^2 C_s^2 \left( k \cdot B \right)^2 / n \right] = 0. \quad (6)$$

Finite wave number vectors $(k \neq 0)$ are allowed in the first and second factors of (6), implying three hyperbolic parts demanding Cauchy data [2]. The last factor may yield both hyperbolic and elliptic ($k = 0$) parts depending on the flow speed. The condition reads rather simple in a two-dimensional system.

Let us assume $\partial_z = 0$. Then, the poloidal components (projections onto the $x$-$y$ plane) of two vectors $V$ and $B$ must align, because the $z$-component of the induction (electron) equation (3) demands

$$\nabla z \cdot (B \times V) + \partial_z \phi_s = \nabla z \cdot (B \times V) = 0, \quad (7)$$

where $\phi_s$ is the scalar potential. We will see that this restriction is the origin of the Alfvén singularity in the ideal MHD equations.
Denoting by $V_p (B_p)$ the poloidal component of the flow velocity (magnetic field), and by $\theta$ the angle between $k$ and $V_p$, the last factor of (6) reads

$$k^4 \cos^2 \theta \left[ V_p^4 \cos^2 \theta - \left( C_s^2 + B^2/n \right) V_p^2 + C_s^2 B_p^2/n \right].$$

Two roots $k \neq 0$ satisfying $\cos^2 \theta = 0$ yield two additional hyperbolic parts. Using “Casimir invariants” associated with the Poisson-bracket structure of the two-dimensional MHD equations, we can easily integrate the total five hyperbolic parts of the system by giving the Cauchy data [4]. Remaining two roots are rather complex. If $0 < (C_s^2 + B^2/n)V_p^2 - C_s^2 B_p^2/n < V_p^4$, we obtain real roots, implying that the remaining parts are also hyperbolic. Denoting the speed of the fast and slow waves by $V_{f,s}^2 = \left( C_s^2 + B^2/n \right) \pm \sqrt{(C_s^2 + B^2/n)^2 - 4C_s^2 B_p^2/n}/2$, and the cusp speed by $V_c^2 = C_s^2 B_p^2/(nC_s^2 + B^2)$, we find that the remaining equations are hyperbolic if $V_c^2 < V_p^2 < V_s^2$ or $V_f^2 < V_p^2$ (hyperbolic PDEs may have discontinuous solutions, describing standing shocks [6]). Otherwise, we obtain elliptic PDEs, which, however, contain the Alfvén singularity.

### C. Incompressible flowing MHD equilibrium equation and the Alfvén singularity

To highlight the Alfvén singularity, let us consider an incompressible flow, and eliminate the hyperbolic transition. Taking the limit of $C_s^2 \to \infty$, we obtain $V_c^2, V_s^2 \to B_p^2/n$ and $V_f^2 \to \infty$, and the characteristics equation (6) reduces to

$$k^2 \left[ k \cdot V \right] \left[ k \cdot \left( V + B/\sqrt{n} \right) \right]^2 \left[ k \cdot \left( V - B/\sqrt{n} \right) \right]^2 = 0. \quad (8)$$

The last factor of (6) has changed into the elliptic parts ($k^2 = 0$) and the Doppler-shifted Alfvén waves. The vanishing of the Doppler-shifted Alfvén waves corresponds to the Alfvén singularity [8, 9]. The incompressible version of the ideal MHD equilibrium equation reads

$$(1 - n\phi'^2)\Delta \psi - \left( n\phi'^2 \right)' \left| \nabla \psi \right|^2 / 2 + G(\psi)' = 0, \quad (9)$$

where $\psi$ is the flux function of the magnetic field, $\phi$ is the stream function of the flow velocity, $G(\psi)$ is a function of $\psi$ and the prime indicates the derivative with respect to $\psi$ (see the next section for the derivation). When $n\phi'^2 = 1$ (equivalent to $V_p^2 = B_p^2/n$), the principal part of the differential equation (9) vanishes. This is the Alfvén singularity.

We note that the Alfvén singularity is related with the singular eigenvalue problem of Alfvén waves. In the linear wave theory, one may analyze the frequency $(\omega)$ spectrum of the
linearized MHD equations. The Alfvén wave has a singular eigenfunction belonging to the continuous spectrum. In flowing plasmas, the Alfvén continuum may Doppler-shift down to the zero frequency. Then, the zero-ω Alfvén resonance yields the singularity in the static (zero-ω) linearized equation. The fully nonlinear version of this singular equation is (9).

III. SINGULAR PERTURBATION INDUCED BY THE HALL EFFECT

A. Hall-MHD equilibrium equation

We add the Hall term to the ideal MHD equations. Here, we consider an incompressible stationary plasma; the governing equations are

\begin{align}
(V \cdot \nabla) V - (\nabla \times B) \times B + \nabla p &= 0, \quad (10) \\
(V - \varepsilon \nabla \times B) \times B + \nabla (\varepsilon p_e - \phi_s) &= 0, \quad (11) \\
\nabla \cdot B &= 0, \quad (12) \\
\nabla \cdot V &= 0, \quad (13)
\end{align}

where \( p_e \) is the electron pressure normalized by \( B_0^2/\mu_0 \). The scaling coefficient \( \varepsilon = \delta_i/L_0 \) measures the ion skin depth \( \delta_i = c/\omega_{pi} = \sqrt{M/\mu_0 ne^2} \) (the constant density is assumed). We assume that \( \varepsilon \) is a small parameter. The terms multiplied by \( \varepsilon \) are the additions to the standard MHD model. Since the Hall term contains the higher order derivative, it works as a singular perturbation to the ideal MHD system [13]. While the ideal MHD is a scale-less model (the scale transform does not change the equation), the Hall-MHD model has an intrinsic length scale, i.e., the ion skin depth.

The singular perturbation due to the Hall effect changes the principal part of the governing PDEs. The characteristic equation of the system (10)-(13) becomes

\[ k^4(k \cdot V)^2(k \cdot B)^2 = 0. \] \hspace{1cm} (14)

In contrast to (8), (14) has no singularity (excepting \( V = 0 \) and \( B = 0 \)). Equation (14) shows that the system of the Hall-MHD equilibrium equations consists of four elliptic and four hyperbolic PDEs. The Cauchy characteristics are the streamlines of \( V \) and \( B \).
B. Two-dimensional system and canonical structures

The integration of the hyperbolic parts is almost straightforward in a two-dimensional system because of the canonical structures of the system. We assume translation symmetry \( \partial_z = 0 \). We may write \( B \) and \( V \) in the forms of

\[
B(x, y) = \nabla \psi(x, y) \times \nabla z + B_z(x, y) \nabla z, \tag{15}
\]

\[
V(x, y) = \nabla \phi(x, y) \times \nabla z + V_z(x, y) \nabla z, \tag{16}
\]

where \( \psi \) (\( \phi \)) and \( B_z \) (\( V_z \)) are the flux (stream) function and z-component of the magnetic field (flow velocity). We denote the poloidal \((x-y)\) components of variables by putting subscript \( p \); for example, \( B_p = \nabla \psi \times \nabla z \). The current density \( j \) is calculated as

\[
j = \nabla \times B = \nabla B_z \times \nabla z - \Delta \psi \nabla z,
\]

where \( \Delta = \partial_x^2 + \partial_y^2 \) is the two-dimensional Laplacian. Substitution of (15) and (16) into (10) and (11) yields

\[
-\Delta \psi \nabla \psi + \Delta \phi \nabla \phi - \nabla G + \{\psi, B_z\} - \{\phi, V_z\} \nabla z = 0, \tag{17}
\]

\[
(\epsilon \Delta \psi + V_z) \nabla \psi - B_z \nabla (\phi - \epsilon B_z) + \nabla (\epsilon p_x - \phi_s) + \{\psi, \phi - \epsilon B_z\} \nabla z = 0, \tag{18}
\]

where \( G = p + V_p^2/2 + B_z^2/2 \) and the Poisson bracket \( \{,\} \) is defined by \( \{P, Q\} = (\partial_y P)(\partial_x Q) - (\partial_x P)(\partial_y Q) \).

C. Casimir invariants associated with the electron momentum balance

We first integrate the hyperbolic parts included in the electron momentum balance equation (18). From the “toroidal” \((z)\) component of (18), we obtain a Casimir invariant

\[
\phi - \epsilon B_z = Y_1(\psi), \tag{19}
\]

where \( Y_1(\psi) \) is an arbitrary function of \( \psi \). Giving \( Y_1(\psi) \) as a Cauchy data, the corresponding hyperbolic part of the system has been integrated.

Equation (19) implies that \( \psi \) and \( \phi \) can be independent functions (the contours of \( \psi \) and \( \phi \) do not coincide). This is an essential difference from the ideal MHD equilibrium; if \( \epsilon = 0 \), (19) reduces to \( \phi = Y_1(\psi) \), demanding that \( \phi \) must be a function of \( \psi \). The Hall effect allows that the two characteristics, the streamlines of \( V_p \) and \( B_p \), take different orbits.
Substitution of (19) into the poloidal component of (18) yields the second Casimir invariant
\[ \varepsilon p_e - \phi_s = Y_2(\psi). \] (20)

Using (19) and (20), (18) reads
\[ (\varepsilon \Delta \psi + V_z - B_z Y_1' + Y_2') \nabla \psi = 0, \] (21)
where the prime indicates the derivative with respect to \( \psi \). We note that the Casimir invariants (19) and (20) complete the Cauchy data associated with \((k \cdot B)^2 = 0\).

D. Analysis of the ion momentum balance – two branches of equilibria

It remains to integrate the momentum balance equation (17). Using (19), we write \( \phi = Y_1(\psi) + \varepsilon B_z \), and substitute it into the poloidal component of (17);
\[ - \left[ \left( 1 - Y_1'^2 \right) \Delta \psi - \left( Y_1'^2 \right) |\nabla \psi|^2 / 2 - \varepsilon Y_1' \Delta B_z \right] \nabla \psi \]
\[ + \varepsilon \left( Y_1'' \Delta \psi + Y_1''' |\nabla \psi|^2 + \varepsilon \Delta B_z \right) \nabla B_z - \nabla G = 0. \] (22)

If we set \( \varepsilon = 0 \) in (22), we must assume \( G = G(\psi) \), and we obtain the ideal MHD equilibrium equation (9) [the other relations \( \phi = \phi(\psi), \phi_s = \phi_s(\psi) \) and \( V_z = \phi' B_z + \phi'_s \), respectively, are derived from (19), (20) and (21) with \( \varepsilon = 0 \)].

In the Hall-MHD system (\( \varepsilon \neq 0 \)), we can consider two different branches of equilibria – one is singular and the other is regular. The two-dimensional vector relation (22) may be satisfied in two different ways; one is the case when all the vectors on the left-hand side of (22) align in the direction of \( \nabla \psi \) and the coefficient multiplying \( \nabla \psi \) vanishes, and the other is the case when the left-hand side consists of two linear independent components (that may be decomposed in terms of \( \nabla \psi \) and \( \nabla \phi \)) and both coefficients vanish simultaneously.

Here we remark that the second branch is overlooked in one-dimensional theories, because all variations of fields can occur only in a single direction in one-dimensional systems.

The first branch assumes \( \phi = \phi(\psi) \), i.e., the poloidal flow aligns the poloidal magnetic field. Then, (19) implies \( B_z = B_z(\psi) \), and (22) becomes
\[ - \left\{ \left[ 1 - (Y_1' + \varepsilon B_z)^2 \right] \Delta \psi - \left[ (Y_1' + \varepsilon B_z)^2 \right] |\nabla \psi|^2 / 2 + G' \right\} \nabla \psi = 0. \] (23)
Vanishing of the coefficient reads as the equilibrium equation. This branch of equilibrium is singular at the point of \((Y_1' + \varepsilon B_z')^2 = 1\). We observe that the ideal MHD equilibrium \((\varepsilon = 0)\) is included in this branch.

For the second branch, we assume that \(\phi\) is not a function of \(\psi\), i.e., the poloidal flow velocity is not parallel to the poloidal magnetic field. By (19), we find that this branch is available only when \(\varepsilon \neq 0\). The two-dimensional vector relation (22) yields two simultaneous equations demanding that the coefficients of two independent vectors vanish. To take \(\nabla \psi\) and \(\nabla \phi\) to be the two independent vectors, we go back to (17). Calculating \(\varepsilon \times (17)+(18)\), we obtain the ion momentum equation

\[
(\varepsilon \Delta \phi - B_z) \nabla \phi + V_z \nabla (\psi + \varepsilon V_z) - \nabla \left( \varepsilon p_i + \varepsilon V^2/2 + \phi_s \right) - \{\phi, \psi + \varepsilon V_z\} \cdot \nabla z = 0, \tag{24}
\]

where the ion pressure is written as \(p_i = p - p_e\). From the toroidal component of (24), we obtain the third Casimir invariant

\[
\psi + \varepsilon V_z = F_1(\phi). \tag{25}
\]

Substituting (25) into the poloidal component of (24), we get the fourth (last) Casimir invariant

\[
\varepsilon p_i + \varepsilon V^2/2 + \phi_s = F_2(\phi). \tag{26}
\]

Using (25) and (26), (24) reads

\[
\left( \varepsilon \Delta \phi - B_z + \varepsilon \dot{F}_1 - \dot{F}_2 \right) \nabla \phi = 0, \tag{27}
\]

where the dot indicates the derivative with respect to \(\phi\). The Casimir invariants (25) and (26) correspond to the Cauchy data associated with \((k \cdot V)^2 = 0\). Hence, the hyperbolic parts of the original equations [see (14)] have been already integrated. From the Casimir invariants (19) and (25), we may recover the physical quantities

\[
B_z = [\phi - Y_1(\psi)]/\varepsilon, \tag{28}
\]

\[
V_z = [F_1(\phi) - \psi]/\varepsilon. \tag{29}
\]

The remaining two elliptic parts of the system read as simultaneous two Poisson equations governing \(\psi\) and \(\phi\); substituting (28) and (29) into (21) and (27) yields

\[
-\varepsilon^2 \Delta \psi = -\psi + Y_1 Y_1' - Y_1' \phi + F_1 + \varepsilon Y_2', \tag{30}
\]

\[
-\varepsilon^2 \Delta \phi = -\phi + \dot{F}_1 - \dot{F}_2 \psi + Y_1 - \varepsilon \dot{F}_2. \tag{31}
\]

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We have to give boundary conditions for \( \psi \) and \( \phi \). For example, we assume \( \mathbf{n} \cdot \mathbf{B} = 0 \) and \( \mathbf{n} \cdot \mathbf{V} = 0 \) on the boundary (\( \mathbf{n} \) is the unit normal), which are equivalent to \( \psi = \text{constant}(= 0) \) and \( \phi = \text{constant}(= 0) \) on the boundary.

We note that the system of equations (30)-(31) is a reduced version of the two-fluid equilibrium equations derived, for example, in Ref. [17]. In the present formulation, simplifications are due to the omission of the compressibility and the electron inertia.

In the two-fluid system, the compressibility causes the hyperbolicity, the condition of which reads \( M^2 \geq 1 \) or \( M^2_i \geq 1 \) where \( M_e \) (\( M_i \)) is the poloidal sonic Mach number for the electron (ion) fluid [17]. This condition is similar to that of the neutral fluid, and is in contrast to that of the MHD system that has the Cauchy characteristics with the Doppler-shifted waves (see Sec. II B). In the Hall-MHD system, the condition reads \( M^2_i \geq 1 \) as with the two-fluid case [14] (the omission of electron inertia eliminates the condition for electron fluid), and the omission of the compressibility eliminates the hyperbolicity, resulting in the elliptic equations (30)-(31).

If the electron inertia is taken into account, the two-fluid system has three characteristic functions [17], viz., \( \psi \), \( \phi \) and \( \psi_e \) (\( \phi_e \) is the stream function of electron fluid). In the Hall-MHD system, the omission of the electron inertia limits \( \phi_e \) to a function of \( \psi \), yielding two different characteristics (contours of \( \psi \) and \( \phi \)). Without the Hall effect, the MHD system restricts the two characteristics as \( \phi = \phi(\psi) \) (see Sec. II B), which leads to the Alfvén singularity.

IV. NUMERICAL ANALYSIS OF THE SINGULAR PERTURBATION

A. Setting of the problem

We solve the regularized Hall-MHD equilibrium equations (30) and (31) with changing \( \varepsilon \) (0.2 < \( \varepsilon \) < 3.2). The Alfvén singularity (of the ideal limit) is arranged to occur inside the domain \( D = [0, 1] \times [0, 1] \). We use the finite difference method with 200 x 200 grid points.

To simplify the analysis, we assume homogeneous distributions of the energy densities, and set \( Y_2(\psi) = \text{constant} \) and \( F_2(\phi) = \text{constant} \). The inhomogeneities in \( B_z \) and \( V_z \) play important roles here; see (28) and (29). We assume

\[ Y_1(\psi) = C_1 + a_1 \psi + a_2 \psi^2, \]
\[ F_1(\phi) = C_2 + b_1 \phi + b_2 \phi^2, \]

where \( a_j, b_j \) and \( C_j \) \((j = 1, 2)\) are arbitrary constants. We relate \( C_j \) to the “toroidal fluxes” of the current and the vortex, and control the latter ones: the total toroidal current \( I_t = \int_D j_z dS \) and the total toroidal vorticity \( \Omega_t = \int_D (\nabla \times \mathbf{V})_z dS \). Giving appropriate parameters \((a_1 = 2 \times 10^{-5}, a_2 = 1, b_1 = b_2 = 10^{-5}, I_t = 10 \) and \( \Omega_t = 9\)), we solve (30) and (31) as a boundary value problem. We assume \( \psi = \phi = 0 \) on the boundary of \( D \).

**B. Measures of the singular perturbation**

We introduce two quantities as measures of the singular perturbation (Hall effect), i.e., they detect the difference between the ideal MHD and Hall-MHD equilibria.

The first one is the angle \( \theta_1 \) between the poloidal magnetic field and the poloidal flow velocity, which is evaluated by

\[
\sin^2 \theta_1 = \frac{|\mathbf{B}_p \times \mathbf{V}_p|^2}{(|\mathbf{B}_p|^2 |\mathbf{V}_p|^2)}.
\]

In the ideal MHD equilibrium, only \( \theta_1 = 0 \) or \( \theta_1 = \pi \) is possible; see (7). Hence, the deviation of \( \sin \theta_1 \) from 0 indicates the action of the Hall effect.

The second measure is the angle \( \theta_2 \) between the poloidal magnetic field and the poloidal current, which is evaluated by

\[
\sin^2 \theta_2 = \frac{|\mathbf{B}_p \times \mathbf{j}_p|^2}{(|\mathbf{B}_p|^2 |\mathbf{j}_p|^2)}.
\]

In the ideal MHD equilibrium, we have \( B_z = B_z(\psi) \) and \( \mathbf{j}_p = \nabla B_z \times \nabla z \), and hence, only \( \theta_2 = 0 \) or \( \theta_2 = \pi \) is allowed. As textbooks teach, the current flowing perpendicular to the magnetic field is a direct measure of the Hall effect.

We denote by \( \langle \rangle \) the area-averaging, i.e., \( \langle f \rangle = \int_D f dS \).

**C. Regularization of the singularity**

In Fig. 1, we plot \( \langle \sin^2 \theta_1 \rangle \) as a function of \( \varepsilon \). We observe that \( \langle \sin^2 \theta_1 \rangle \) becomes small as \( \varepsilon \) reduces, implying that the poloidal flow velocity is almost parallel to the poloidal magnetic field in the major region when \( \varepsilon \) is sufficiently small. We, thus, see that the global structure may be approximated by the ideal MHD model.
All these solutions contain the “singular points” \( Y_1^{(2)} = 1 \) (\(|Y_1^{(2)}|\) is the “poloidal Alfvén Mach number” in the MHD limit). In the neighborhood of the singular points, the Hall effect plays its role to “heal” the singularity. Figure 2 shows \( \langle \sin^2 \theta_2 \rangle \) as a function of \( \varepsilon \). We observe that \( \langle \sin^2 \theta_2 \rangle \) begins to increase when \( \varepsilon \) is reduced less than 0.24. This is due to the local amplification of the perpendicular current near the singular points.

In Fig. 3 we plot \( \sin^2 \theta_2 \) as a function of \( Y_1^{(2)} \) (\( \varepsilon = 0.207 \)). The Hall effect becomes large when \( Y_1^{(2)} \) is close to 1.

Figure 4 is the contour plot of \( \sin^2 \theta_2 \) superimposed by the vector field of the poloidal current \( j_p \) (\( \varepsilon = 0.207 \)). In the neighborhood of the singular points, \( \sin^2 \theta_2 \) approaches to 1, i.e., \( j_p \) becomes perpendicular to \( B_p \). Across the singular points, the parity of \( \text{sgn}(j_p \cdot B_p) \) flips; the change of the parity occurs “smoothly” with rotating \( j_p \). This shows the mechanism how the Hall effect heals the Alfvén singularity. In the ideal MHD model, \( j_p \) must be always parallel to \( B_p \). The Alfvén singularity separates the domains with different parities of \( \text{sgn}(j_p \cdot B_p) = \text{sgn} \cos \theta_2 \). To cross the domain boundary with keeping \( j_p \) parallel to \( B_p \), \( j_p \) must vanish (contradicting with other relations) at the boundary. This causes the Alfvén singularity. The Hall effect, removing the restriction binding \( j_p \) and \( B_p \), allows smooth connection of the different domains.

Figure 5 shows the spatial profiles of \( \theta_2 \), \( |B_p| \) and \( |j_p| \) on a cut at \( y = 0.6 \). The angle \( \theta_2 \) switches from 0 (parallel) to \( \pi \) (anti-parallel), while \( |j_p| \) keeps a finite magnitude.

V. DISCUSSION

The ideal MHD model falls short of describing the flowing plasma equilibrium with velocity around the Alfvénic speed. To investigate the structure of fields near the Alfvén singularity, we have to invoke a higher order effect, that is, in a collision-less plasma, the Hall effect. The Hall-MHD model is no longer scale-free – the ion skin depth determines the minimum length scale of the field variation, at which the Hall term (a singular perturbation) can balance with the other macroscopic (ideal MHD) terms and heals the singularity. Here, the Alfvén singularity, caused by the macroscopic field distribution, set a stage for the small-scale Hall effect. We have formulated the determining equations (30)-(31), and have seen how the Hall effect plays its role to cure the Alfvén singularity (Figs. 4 and 5).

In the Hall-MHD systems, two vectors \( B_p \) and \( j_p \) may take different directions. The ideal
MHD limit ($\varepsilon \rightarrow 0$), however, binds these two vectors to be parallel or anti-parallel. Two domains having different parities of $\text{sgn}(\mathbf{j}_p \cdot \mathbf{B}_p)$ are, then, separated by the Alfvén singularity. With the help of the Hall effect, $\mathbf{j}_p$ can smoothly rotate with respect to $\mathbf{B}_p$ through the boundary of the two different domains. We note that such a regularized solution must be “two-dimensional.” Indeed, even if we can find one-dimensional regular solutions satisfying (30)-(31) such as $\psi = \psi(r)$ and $\phi = \phi(r)$ with $r = \sqrt{x^2 + y^2}$, these are special solutions satisfying the singular equation (23) since the coefficient of (23) vanishes by virtue of the Casimir invariants and (30)-(31).

Viewing the macroscopic scale from the Hall-MHD framework, how can we describe the global distribution of the fields? When $\varepsilon$ is small, one may expect that the macroscopic fields are well approximated by the ideal MHD model, excepting the singular points. This is, however, a subtle question, because the ideal MHD model is based on the \textit{a priori} assumption that the fields vary only at a large ($\varepsilon \ll 1$) length scale. In our calculations, we have seen that the Hall-MHD solutions converge to ideal MHD solutions almost everywhere excepting the transition region, the $\varepsilon$-neighborhood of the Alfvén singularity. We have pointed out that the regularized solutions of the Hall-MHD and the singular MHD-like solutions belong to different branches of equilibria (Sec. III D). Outside the transition region, both branches merge with reducing $\sin^2 \theta_1$ (Fig. 1).

This observation, however, does not deny the possibility of small-scale fields to fill up a wide space. In the calculations given in Sec. IV, we assumed relatively small parameters $a_j$ and $b_j$. When we increase these parameters, the solutions oscillating at the length scale of $\varepsilon$ bifurcate from the smooth solutions of the nonlinear Poisson system (30)-(31); the double Beltrami field [13] is an example of such solutions.

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Figure captions

FIG. 1: Semi-log plot of $\langle \sin^2 \theta_1 \rangle$ as a function of $\varepsilon$, where $\theta_1$ is the angle between the poloidal magnetic field and the poloidal flow velocity, the bracket indicates the area-averaging, and $\varepsilon$ is the measure of the ion skin depth. For small $\varepsilon$, the poloidal flow velocity is globally parallel to the poloidal magnetic field.

FIG. 2: Semi-log plot of $\langle \sin^2 \theta_2 \rangle$ as a function of $\varepsilon$, where $\theta_2$ is the angle between the poloidal magnetic field and the poloidal current density. Other symbols are the same with Fig.1. For small $\varepsilon$, the perpendicular poloidal current arises.

FIG. 3: Relation between $\sin^2 \theta_2$ and $Y_1'^2$, where $|Y_1'|$ is the poloidal Alfvén Mach number in the MHD limit, and the over-line indicates the averaging for each $Y_1'^2$. The Alfvén singularity in the MHD limit is characterized by $Y_1'^2 = 1$. The effect of the singular perturbation gets large near the singularity.

FIG. 4: Macroscopic structure when $\varepsilon = 0.207$. The gray-scale contour represents $\sin^2 \theta_2$, and the vector field represents the poloidal current. Across the singularity ($\sin^2 \theta_2 \sim 1$), the reversal of the poloidal current occurs.

FIG. 5: Transition across the singularity: one-dimensional plots at $y = 0.6$ when $\varepsilon = 0.207$. The top panel shows the behavior of $\theta_2$, and the bottom one shows the intensity of poloidal fields, $|B_p|$ and $|j_p|$. The angle $\theta_2$ changes from 0 to $\pi$, keeping finite intensities of the fields.
FIG. 1:
FIG. 2:
FIG. 3:
FIG. 4:
FIG. 5: