On transport equations in thin moving domains

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1. Problems on moving surfaces

- Reaction-diffusion on biological cells
- Fluid-flow on biological cell
  Boussinesq (1913), L. E. Scriven (1960)
- Fluid-flow on rotating surface
  geophysics

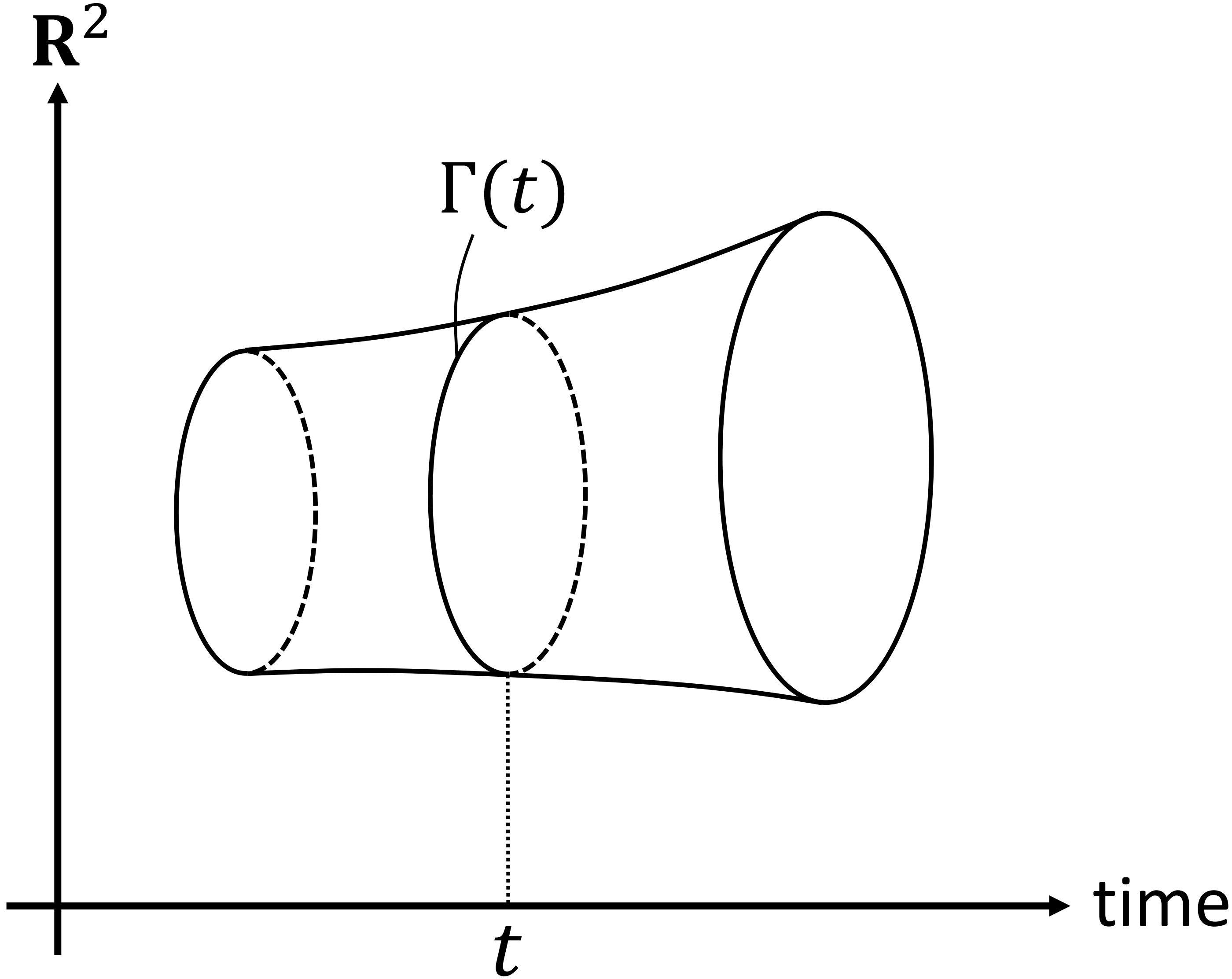
Problems

$\{\Gamma(t)\}_{t\in[0,T)}$: evolving surface in $\mathbf{R}^3$

(1) How does one derive equations describing phenomena on a moving surface $\{\Gamma(t)\}$?

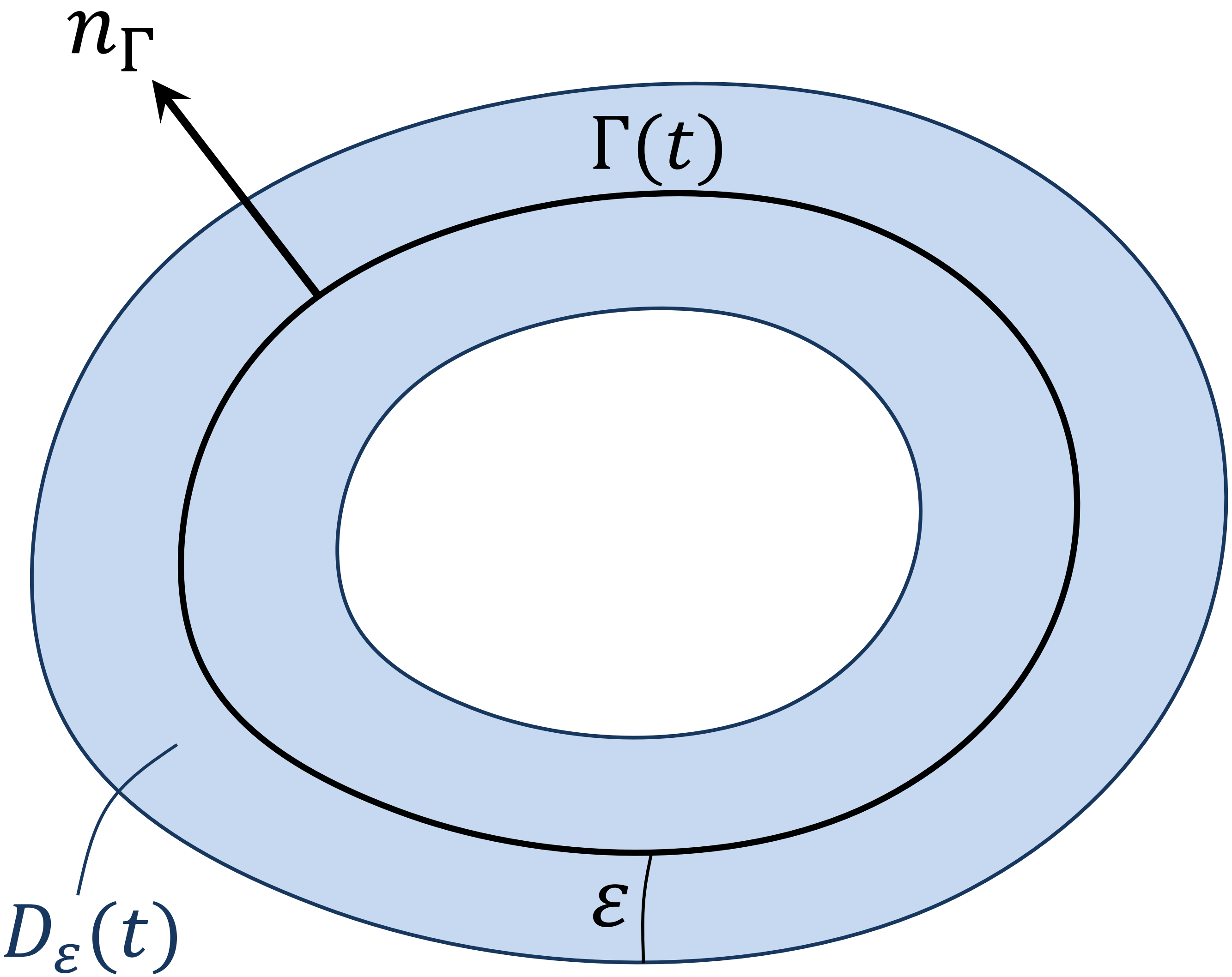
(2) Is it related to zero-width limit of thin moving domain?

$$D_\varepsilon(t) = \left\{\, x \in \mathbf{R}^3 \;\middle|\; \mathrm{dist}\left(x, \Gamma(t)\right) < \varepsilon \,\right\}.$$



$\mathbf{R}^2$
$\Gamma(t)$
time
t

Thin domain



2. Derivations of mass transport equations

2.1 Conservation law

n_Γ : outer unit normal of $\Gamma(t)$

$\eta = \eta(y,t)$: density of substance at $y \in \Gamma(t)$

$v = v(y,t)$: velocity of substance at $y \in \Gamma(t)$

(vector field on $\Gamma(t)$ not necessarily tangential)

Local mass conservation law

V_Γ : normal velocity of $\Gamma(t)$ in the direction of n_Γ.

- Ansatz: Normal component of v equals V_Γ

(There is no flow leaving from or coming to $\Gamma(t)$.)

$$\frac{d}{dt}\int_{U(t)} \eta \, d\mathcal{H}^2 = 0 \text{ for any portion } U(t) \text{ of } \Gamma(t)$$

($U(t)$: relatively open set)

Leibniz formula

$$\frac{d}{dt}\int_{U(t)} \eta \, d\mathcal{H}^2 = \int_{U(t)} (\partial^\bullet \eta + (\mathrm{div}_\Gamma \, v)\,\eta)\, d\mathcal{H}^2$$

$$\partial^\bullet \eta = \partial_t \eta + (v \cdot \nabla)\,\eta$$

(material derivative)

$\mathrm{div}_\Gamma v$: surface divergence

Ansatz implies

$$v = V_\Gamma \, n + v^T.$$

v^T : tangential velocity ($v^T \cdot n_\Gamma = 0$)

Conservation law

The law

$$0 = \frac{d}{dt} \int_{U(t)} \eta \, d\mathcal{H}^2$$

for all $U(t)$ yields

$$\partial^{\bullet} \eta + (\mathrm{div}_{\Gamma} \, v) \, \eta = 0 \quad \text{on} \quad \Gamma(t).$$

Calculation of surface divergence

$$\mathrm{div}_\Gamma v = \mathrm{div}_\Gamma (V_\Gamma\, n) + \mathrm{div}_\Gamma\, v^T$$

$$= V_\Gamma \underbrace{\mathrm{div}_\Gamma\, n}_{-H} + \underbrace{\nabla_\Gamma V_\Gamma \cdot n}_{0} + \mathrm{div}_\Gamma\, v^T$$

H : mean curvature

$$\therefore\ \mathrm{div}_\Gamma\, v = -H\, V_\Gamma + \mathrm{div}_\Gamma\, v^T$$

Note : $\mathrm{div}_\Gamma (\eta\, v^T) = \nabla_\Gamma\, \eta \cdot v^T + \eta\, \mathrm{div}_\Gamma\, v^T$

Second expression of conservation law

Since $\partial^\bullet \eta = \partial^\circ \eta + v^T \cdot \nabla \eta$ with

$$\partial^\circ \eta = \partial_t \eta + V_\Gamma\, n \cdot \nabla \eta,$$

$$\partial^\bullet \eta + (\mathrm{div}_\Gamma\, v)\, \eta$$

$$= \partial^\circ \eta - H\, V_\Gamma \eta + \mathrm{div}_\Gamma(\eta v^T).$$

Mass conservation:

$$\partial^\circ \eta - H\, V_\Gamma\, \eta + \mathrm{div}_\Gamma(\eta v^T) = 0$$

First two terms depend only on motion of $\Gamma(t)$, independent of v^T.

Mass conservation

$$\partial^{\bullet}\eta + (\mathrm{div}_{\Gamma}\, v)\,\eta = 0$$

or

$$\partial^{\circ}\eta - H\, V_{\Gamma}\,\eta + \mathrm{div}_{\Gamma}(\eta v^{T}) = 0$$

$\mathrm{div}_{\Gamma} w$: surface divergence

$$\mathrm{div}_{\Gamma} w = \nabla_{\Gamma} \cdot w$$

$$\nabla_{\Gamma} = P_{\Gamma}\,\nabla$$

$P_{\Gamma} = I - n_{\Gamma} \otimes n_{\Gamma}$: projection to tangent space of $\Gamma(t)$.

2.2 Zero width limit

Mass conservation in a thin domain reads

(CL) $\rho_t + \mathrm{div}(\rho\, u) = 0$ in $D_\varepsilon(t)$

(B) $u \cdot \nu_{D_\varepsilon} = V_{D_\varepsilon}^N$ on $\partial D_\varepsilon(t)$.

The substance in $D_\varepsilon(t)$ moves along the boundary of $D_\varepsilon(t)$. It does not go into or out of $D_\varepsilon(t)$.

$V_{D_\varepsilon}^N$: normal velocity of $\partial D_\varepsilon(t)$ in the direction of unit outer normal ν_D of $D_\varepsilon(t)$

Problem

$$\rho(x,t) = \eta(\pi(x,t),t) + d(x,t)\,\eta^1(\pi(x,t),t) + O(d^2)$$

$$u(x,t) = v(\pi(x,t),t) + d(x,t)\,v^1(\pi(x,t),t) + O(d^2)$$

as $d \to 0$

$\pi(x,t)$: projection to $\Gamma(t)$

$d(x,t)$: signed distance ($d > 0$ near space infinity)

$$\pi(x,t) = x - d(x,t) n_\Gamma(\pi,t)$$

If ρ solves (CL), what equation η solves?

Equation derived from zero width limit

Theorem 1 (T.-H. Miura, C. Liu, Y. G.). Let (ρ, u) solve (CL) in D_ε with (B). Let η, v be the first term of the expansion. Then

$$v = V_\Gamma \, n_\Gamma + v^T$$

and η solves

(CS) $$\partial^\circ \eta - H \, V_\Gamma \, \eta + \mathrm{div}_\Gamma(\eta v^T) = 0.$$

Idea of the proof

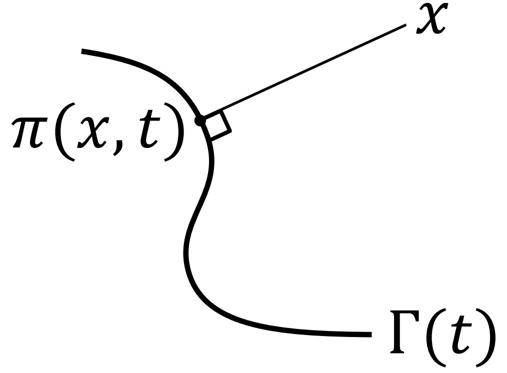
Differentiate

$$\rho(x,t) = \eta(\pi(x,t),t) + d(x,t)\,\eta^1(\pi(x,t),t) + O(d^2)$$

in time.

Note $\partial_t\big(\eta(\pi(x,t),t)\big) = (\partial^\circ\eta)(\pi(x,t),t)$,

$$d_t = -V_\Gamma.$$



We observe

$$\rho_t = (\partial^\circ\eta)(\pi,t) - (V_\Gamma\,\eta^1)(\pi,t) + O(d).$$

Idea of the proof (continued)

Note that

$$\nabla\pi(x,t) = P_\Gamma(\pi,t) + O(d).$$

One observes that

$$(\rho u)(x,t) = (\eta v)(\pi,t) + W^1 d + O(d^2)$$

with $W^1(\pi,t) = (\eta v^1)(\pi,t) + (\eta^1 v)(\pi,t)$

which yields

$$\begin{aligned}\big(\nabla(\rho u)\big)(x,t) &= \nabla\pi(x,t)\nabla(\eta v) + \nabla d \otimes W^1 + O(d)\\ &= (\mathrm{P}_\Gamma \nabla(\eta v))(x,t) + n_\Gamma \otimes W^1 + O(d).\end{aligned}$$

Here $u(x,t) = v(\pi,t) + d(x,t)v^1(\pi,t) + O(d^2)$.

Idea of the proof (continued)

We are now able to conclude

$$\big(\mathrm{div}(\rho u)\big)(x,t) = \mathrm{div}_\Gamma(\eta v) + n_\Gamma \cdot W^1 + O(d).$$

Here

$$\mathrm{div}_\Gamma(\eta v) = -\eta\, V_\Gamma H + \mathrm{div}_\Gamma(\eta v^T).$$

Note $n_\Gamma \cdot W^1 = \eta^1 V_\Gamma$ because $v \cdot n_\Gamma = V_\Gamma$, $v^1 \cdot n_\Gamma = 0$.

Thus

$$\partial_t \rho + \mathrm{div}(\rho u)$$
$$= [\partial^\circ \eta - H\eta V_\Gamma + div_\Gamma(\eta v^T)](\pi, t) + O(d). \square$$

3. Derivation of diffusion equations

3.1 Energy law

$\mathcal{K}$: Kinetic energy

$\mathcal{F}$: Free energy

$$\delta\left(\int_0^T \mathcal{K}\, dt\right) = \int_0^T \int (F_i \cdot \delta x)\, dxdt$$

$$\delta\left(\int_0^T \mathcal{F}\, dt\right) = \int_0^T \int (F_c \cdot \delta x)\, dxdt$$

δ : variation with respect to flow map

Least action principle LAP

F_i : inertial force

F_c : conservative force

Stationary point of the action integral

$$A[x] = \int_0^T \mathcal{K}(x)\,dt - \int_0^T \mathcal{F}(x)\,dt$$

x : Flow map $X \mapsto x(t, X)$

$$\frac{dx}{dt} = u(x,t) \quad x|_{t=0} = X$$

(u : velocity field)

LAP $\quad \frac{\delta A}{\delta x} = 0 \Leftrightarrow F_i - F_c = 0$

Maximum dissipation principle MDP

$\mathcal{D}$: dissipation energy depending on $u = \dot{x}$.

$$\delta\mathcal{D} = F_d \cdot \delta u$$

F_d : dissipation force

LAP + MDP $:\Rightarrow F_i - F_c = F_d$

Diffusion process

$$\mathcal{K} = 0, \quad \mathcal{F}[x] = \int_{D(t)} \omega(\rho(x))\, dx$$

$$\mathcal{D} = \frac{1}{2}\int_{D(t)} \rho|u|^2\, dx \quad \omega : \text{given}$$

ρ : satisfies the transport eq (CL) $\rho_t + \mathrm{div}(\rho u) = 0$

$x : D(0) \to D(t)$ flow map with velocity u

$$x(X, 0) = X, \quad \frac{dx}{dt} = u(x, t)$$

$$X \in D(0)$$

Conservative force

Lemma 1 (T.-H. Miura, C. Liu, Y. G.). Assume that ρ satisfies (CL). Then

$$F_c = \frac{\delta}{\delta x}\mathcal{F} = \nabla p(\rho)$$

with

$$p(\rho) = \omega'(\rho)\rho - \omega(\rho).$$

Conservative force is the pressure gradient.

Diffusion equation

Mass conservation law (CL):

$$\rho_t + \mathrm{div}(\rho u) = 0$$

Darcy's law:

$$\rho u = -\nabla p(p)$$

Example: $\omega(\rho) = \rho \log \rho \Rightarrow p(\rho) = \rho$

$$\rho_t - \Delta\rho = 0 \text{ (heat equation)}$$

Conservation force on a surface

Lemma 2 (MCG). Let (η, v) solves (CS). Then

$$F_c = \frac{\delta \mathcal{F}}{\delta y} = \nabla_\Gamma p(\eta).$$

Here

$$\mathcal{F}(y) = \int_{\Gamma(t)} \omega\big(\eta(y)\big)\, d\mathcal{H}^2(y).$$

Diffusion equations on a surface

Conservation law (CS)

$$\partial^\circ \eta - H\, V_\Gamma\, \eta + \mathrm{div}_\Gamma(\eta v^T) = 0$$

Darcy's law

$$\eta v^T = -\nabla_\Gamma p(\eta)$$

Example: $\omega(\eta) = \eta \log \eta \Rightarrow p(\eta) = \eta$

$$\partial^\circ \eta - H\, V_\Gamma\, \eta - \Delta_\Gamma \eta = 0$$

(different from different equation with convective term Elliot et al.)

3.2 Zero width limit

Consider

$$
\text{(D)} \quad \begin{cases} \rho_t + \mathrm{div}(\rho u) = 0 \\ \rho u = -\nabla p(p) \quad \text{in} \quad D_\varepsilon(t) \\ p(\rho) = \omega'(\rho)\rho - \omega(\rho) \end{cases}
$$

with (B) $u \cdot \nu_\Omega = V_\Omega^N$ on $\partial D_\varepsilon(t)$.

Expanded pressure $p(\rho(x,t)) = p^0(\pi,t) + d(x,t)p^1(x,t) + O(d^2)$

Equation derived from zero width limit

Theorem 2 (MLG). Let (p, u) solves (D) in $D_\varepsilon(t)$ with (B). Let (η, v) be the first term of the expansion. Then

$$\partial^\circ \eta - H\, V_\Gamma\, \eta + \mathrm{div}_\Gamma(\eta v^T) = 0$$

$$\eta v^T = -\nabla_\Gamma p^0 \quad \text{on} \quad \Gamma(\mathrm{t}).$$

Energy identity

For diffusion equations (D) on $D(\mathrm{t})$ with (B), we have

$$\frac{d}{dt}\int_{D(t)} \omega(\rho)\,dx$$

$$= -\int_{D(t)} \rho|u|^2\,dx - \int_{\partial D(t)} p(\rho)V_{\Omega}^{N}\,d\mathcal{H}^2$$

i.e.

$$\frac{d\mathcal{F}}{dt} = -2\mathcal{D} + \dot{W},$$

$$\dot{W} = -\int_{\partial D(t)} p(\rho)V_{D}^{N}\,d\mathcal{H}^2.$$

$\dot{W}$: rate of change of work by the eternal environment

Energy identity on a surface

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^2$$

$$= -\int_{\Gamma(t)} \eta |v^T|^2 \, d\mathcal{H}^2 + \int_{\Gamma(t)} p(\eta) V_\Gamma H \, d\mathcal{H}^2,$$

$$\dot{W} = -\int_{\Gamma(t)} p(\eta) V_\Gamma H \, d\mathcal{H}^2.$$

Commutativity

Diffusion equation in $D_\varepsilon(t)$ $\xrightarrow{\text{integration by parts}}$ Energy identity in $D_\varepsilon(t)$

Energy identity in $D_\varepsilon(t)$ $\xrightarrow{\text{energetic variation}}$ Diffusion equation in $D_\varepsilon(t)$

Diffusion equation in $D_\varepsilon(t)$ ↓ $\varepsilon \to 0$

Energy identity in $D_\varepsilon(t)$ ↓ $\varepsilon \to 0$ (MLG)

Diffusion equation in $\Gamma(t)$ $\xrightarrow{\text{integration by parts}}$ Energy identity in $\Gamma(t)$

Energy identity in $\Gamma(t)$ $\xrightarrow{\text{energetic variation}}$ Diffusion equation in $\Gamma(t)$

Rigorous results

- If $\Gamma(t)$ does not move, a solution of reaction diffusion equation in D_ε tends to a solution of equation on Γ in some rigorous sense in various settings. (J. Hale, G. Raugel, 1992)……
- However, for moving $\Gamma(t)$, the first convergence result is given by T.-H. Miura (2016, IFB to appear). (heat eq)
- For existence of a solution for the heat equation on moving surface with **convective term** (A. Reusken et al.)

4. Derivation of the Euler and the Navier-Stokes equations

H. Koba, C. Liu, Y. G. (2016)

Incompressible Euler and Navier-Stokes on moving hypersurface: Energetic derivation

H. Koba, Compressible flow

Energetic formulation (Euler)

$$\mathcal{K}[x] = \int_{\Gamma(t)} \frac{\eta |v|^2}{2} dx$$

η : density

$$\mathrm{div}_{\Gamma} v = 0 \quad \text{(incompressibility)}$$

LAP⇒ $\delta \int_0^T \mathcal{K}[x]\, dt = 0 \quad \text{under} \quad \mathrm{div}_{\Gamma} v = 0$

Euler equation on $\Gamma(t)$

$$\partial^\bullet \eta = 0$$

(I) $\eta \partial^\bullet v + \operatorname{grad}_\Gamma \sigma + \sigma H n = 0$

$$\operatorname{div}_\Gamma v = 0$$

(This is overdetermined.)

(II) $\partial^\bullet \eta = 0$

$$P_\Gamma(\eta \partial^\bullet v) + \operatorname{grad}_\Gamma \sigma = 0$$

$$\operatorname{div}_\Gamma v = 0$$

(Variation is taken only in tangential direction.)

Dissipative energy

$$\mathcal{D} = \mu \int_{\Gamma(t)} |D_\Gamma(v)|^2 \, d\mathcal{H}^2, \quad \mu > 0$$

$$D_\Gamma(v) = P_\Gamma D(v) P_\Gamma$$

$$D(v) = \frac{1}{2}\left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^i}{\partial x_1}\right)$$

(projected strain rate)

Navier-Stokes equations

$$\partial^\bullet \eta = 0$$

(III) $$\eta \partial^\bullet v + \mathrm{grad}_\Gamma\, \sigma + \sigma H n = 2\mu\, \mathrm{div}_\Gamma D_\Gamma(v)$$

$$\mathrm{div}_\Gamma v = 0$$

(overdetermined)

$$\partial^\bullet \rho = 0$$

(IV) $$\eta P_\Gamma \partial^\bullet v + \mathrm{grad}_\Gamma\, \sigma = 2\mu P_\Gamma \mathrm{div}_\Gamma D_\Gamma(v)$$

$$\mathrm{div}_\Gamma v = 0$$

Relation to other derivation

Incompressible perfect flow

V. I. Arnold (1966) $\eta = \mathrm{const}$

$$\eta P_\Gamma(\partial^\bullet v) + \mathrm{grad}_\Gamma \sigma = 0$$

for a fixed surface

(variational principle on the space of volume preserving diffeormorphism)

(Least action principle)

Incompressible viscous flow on a fixed surface

M. E. Taylor (1992)

Balance of momentum on manifold

D. Bothe, J. Prüss (2010)

Boussinesq-Scriven model

T. Jankuhn, M. A. Olshanskii, A. Reusken

Preprint (2017)

Zero width limit

$$\partial_t u + (u, \nabla) u + \nabla p = \mu_0 \Delta u$$
$$\operatorname{div} u = 0 \quad \text{in} \quad D_\varepsilon(t)$$

Boundary condition (perfect slip)

$$u \cdot \nu_D = V_D^N$$

$$(D(u)\nu)_{\mathrm{tan}} = 0$$

Limit equation

Result (T.-H. Miura)

$$P_\Gamma(\partial_t v + (v,\nabla)v) + \nabla_\Gamma q$$
$$= 2\mu_0 P_\Gamma \mathrm{div}(P_\Gamma D(v) P_\Gamma)$$

This agrees with what KCG derived.

Summary

We derive equations based on transport of mass on a moving surface by several methods

- energetic variational principle
- zero width limit of equations on a moving thin domain