Putting the D in MRMHD

a prescription for all that ails ideal MHD!

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Phys. Plasmas: accepted March 10, 2017!

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Why and how to fix Ideal MHD?

- Ideal MHD is overconstrained
  - No heat transport along field lines
  - No reconnection so islands or chaos cannot form
  - Thus *inapplicable* to hot and 3D plasmas!

- Fix by removing the bad constraints and keeping the good, doing more with less!
MRxMHD: M stands for **Multi-region** (aka waterbag)
Rx stands for **Relaxed**; ..D stands for **Dynamics**

Fundamental postulates of new general reformulation of MHD:

- \( \exists \) transport interfaces, \( I_i \) or \( \Gamma_{i,j} \), or \( \partial \Omega_{i,j} \) (e.g. nested tori or island separatrices), that act like sheets of ideal-MHD plasma

- Plasma relaxes (in some generalized Taylor sense) in regions \( \mathcal{P}_i \) (or \( \Omega_i \)) bounded by the interfaces

- Only a *subset* of ideal-MHD invariants apply
SPEC (currently) uses MRxMHS, not MRxMHD:

MRxMHS = Multi-region Relaxed MagnetohydroStatic$ics$ (i.e. equilibrium theory)

• Taylor relaxation *energy* principle
• constant *pressure* in each region

MRxMHD = Multi-region Relaxed MagnetohydroDynamic$ics$

*New approach:* use *Hamilton’s Principle* — stationarity of time-integrated *Lagrangian*

✦ constant *temperature* in each region
✦ supports sound waves within relaxation regions as well as radially compressible and Alfvén modes + *tearing*
✦ can treat *development* of resonant current sheets
✦ can add equilibrium flow to SPEC and will be basis for a new time-evolution waterbag code

Ref. Stuart Hudson’s talk yesterday
MRxMHD Lagrangian is *kinetic energy* minus MHD *potential energy* + constraint terms:

- MHD Lagrangian density in region $i$
  \[
  \mathcal{L}^{\text{MHD}} = \rho \frac{v^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}
  \]

- Constrained Lagrangian in region $i$
  \[
  L_i = \int_{\Omega_i} \mathcal{L}^{\text{MHD}} dV + \tau_i (S_i - S_{i0}) + \mu_i (K_i - K_{i0})
  \]

- Helicity and entropy *macroscopic* invariants
  \[
  K_i \equiv \int_{\Omega_i} \frac{\mathbf{A} \cdot \mathbf{B}}{2\mu_0} dV \quad S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln \left( \kappa \frac{p}{\rho^\gamma} \right) dV
  \]
In varying action, \( \rho \) is constrained \textit{holonomically} to the displacement \( \xi \) of each fluid element:

- Mass conserved \textit{microscopically}, i.e. pointwise
  \[
  \delta \rho = -\nabla \cdot (\rho \xi) \text{ in } \Omega_i
  \]

- Helicity and entropy constrained \textit{macroscopically}, throughout \( \Omega_i \), using Lagrange multipliers \( \mu_i \) and \( \tau_i \), while \( p \) and \( A \) are free fields

- Including vacuum field energy, total Lagrangian is
  \[
  L = \sum_i L_i - \int_{\Omega_v} \frac{B \cdot B}{2\mu_0} \, dV
  \]

- Setting variation of action to 0 gives EL equations:
  \[
  \delta \int L \, dt = 0
  \]
Equations within $\Omega_i$

- Mass conservation (microscopic constraint)
  \[ \frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \mathbf{v}) \]

- $\delta \rho \Rightarrow$ Isothermal equation of state
  \[ p = \tau_i \rho \quad (N.B. \quad \tau_i = C_{si}^2) \]

- $\delta \mathbf{A} \Rightarrow$ Beltrami equation
  \[ \nabla \times \mathbf{B} = \mu_i \mathbf{B} \quad (N.B. \Rightarrow j \times \mathbf{B} = 0) \]

- $\xi \Rightarrow$ Momentum equation (Euler fluid)
  \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p \]
Equations on interface $\Gamma_{i,j}$

- $\xi \Rightarrow$ Force balance
  \[ p + \frac{B^2}{2\mu_0} \bigg|_{i,j} = 0 \]

- Surface constraints
  \[ n_i \cdot B = 0 \quad \text{on } \partial \Omega_i \]
  \[ n_i \cdot [v]_{i,j} = 0 \quad \text{on } \partial \Omega_{i,j} \]

- Complete set of equations, consistent because derived from single scalar function $L$
Proving the MRxMHD pudding:

- Q1) What is the MRxMHD spectrum and what are the effects of field-line curvature and equilibrium mass flow on stability?

- Q2) When are the current sheets topologically stable towards internal plasmoid formation (reconnection)?

- Q3) When do unstable modes saturate at a low level or develop nonlinearly into explosive events?
What happens in static or adiabatic limit?

- $\partial_t \to 0 \implies \nabla \cdot (\rho \mathbf{v}) = 0 \quad \mathbf{v} \cdot \nabla \mathbf{v} = -\tau_i \nabla \ln \rho$

only solutions valid for any flowline configuration, from nested surfaces to arbitrarily chaotic, are

$$\rho = \rho_{0i} \exp \left( -\frac{v^2}{2\tau_i} \right) \implies p = p_{0i} \exp \left( -\frac{v^2}{2\tau_i} \right)$$

(N.B. *incompressible* in limit $v/C_s \to 0$) and

$$\nabla \times \mathbf{v} = \alpha_{0i} \exp \left( -\frac{v^2}{2\tau_i} \right) \mathbf{v}$$

Almost isomorphous to B equation: should be implementable in SPEC. Derivable variationally — N. Sato
Switch on slab boundary ripple to study Resonant Magnetic Perturbations (RMPs). Slow (adiabatic) limit

MRxMHD Hahm-Kulsrud-Taylor (HKT): Rippled Slab Model for resonant current sheets
2-region MRxMHD model*

- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$

Hahm & Kulsrud (HK), Phys. Fluids '85, found 2 solutions:

- shielding current sheet on $x = 0$ (shown in red)

$$\psi = a B^a_y \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

$$\psi = a B^a_y \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where $B^a_y$ is $|\text{unperturbed poloidal field}|$ at boundaries and $\alpha \ll 1$
A good test case for MRxMHD:

- Linearity of Beltrami equation leads to easily solvable, linear GS equation (*Poisson in small-\(\mu\) limit.*)
- Symmetry about, and straightness of, current sheet at \(x = 0\): gives most
- Geometrically simple 2-region relaxation scenario:
  - Switch-on: *ripple* on upper and lower boundaries slowly increased from zero (plane slab) to final amplitude
  - A *shielding current* sheet at \(x = 0\) resonance develops
  - Kruskal-Kulsrud damping: evolution through *equilibria*
  - Connect equilibrium sequence by *helicity conservation*
Grad-Shafranov equation for force-free field in slab geometry:

\[ \mathbf{B} = \nabla z \times \nabla \psi + F(\psi) \nabla z \quad \nabla^2 \psi + FF' = 0 \]

\[ \nabla \times \mathbf{B} = \mu \mathbf{B} \quad \text{(Beltrami equation) is satisfied by requiring:} \]

\[ \nabla^2 \psi = \mu F \quad \text{with} \quad F(\psi) = C - \mu \psi, \quad \text{giving} \quad (\nabla^2 + \mu^2)\psi = C \]

General Solution: \( \psi = \overline{\psi} + \frac{\overline{F}}{B_0} \psi_0(x|\mu) + \hat{\psi}(x, y) \)

where \( \overline{\psi} \) is cross-sectional average of \( \psi \),

\( \psi_0(x|\mu) \equiv \frac{B_0}{\mu} (1 - \cos \mu x) \)

is plane slab solution, \( \overline{F} \) is the cross-sectional average of \( B_z \),

and \( \hat{\psi} \) obeys a homogeneous Beltrami equation \((\nabla^2 + \mu^2)\hat{\psi} = 0 \)

with boundary conditions such that \( \psi \) is constant on boundary and on cuts.
Helicity conservation requires three extensions of HK solution: Instead of the HK harmonic component $\psi_1$ we use ansatz

$$\hat{\psi}(x, y) \equiv \frac{2\alpha \psi_a}{\sinh k_1 a} \left( | \sinh k_1 x | \cos ky \right.$$ 

$$+ \frac{k_1}{\mu} | \sin \mu x | \right) - \overline{\psi} \cos \mu x$$

where:

1. $\hat{\psi}$ is a solution of the *Beltrami equation* $(\nabla^2 + \mu^2)\hat{\psi} = 0$ It is only harmonic in the small-$\mu$ limit. Likewise

   $$k_1(\mu) \equiv (k^2 - \mu^2)^{1/2} \rightarrow k \text{ only as } \mu \rightarrow 0$$

2. The term in $\gamma_S$ was introduced in Dewar *et al.* 2013 to allow control of the *total current* in the sheet

3. The term in $\overline{\psi}$ is required for poloidal flux conservation
• In plane slab, before ripple is turned on, the *unperturbed* equilibrium flux function is

\[ \psi_0(x|\mu_0) \equiv \frac{B_0}{\mu_0} (1 - \cos \mu_0 x) \]

• As amplitude parameter \( \alpha \) is increased from 0, \( \mu \) must *change* to preserve helicity and fluxes:
Current sheet has a strong d.c. component

- HK implicitly assumed the total current in the sheet was zero, but MRxMHD switch-on shows there is a \textit{nonzero} total current $J = \frac{2\alpha \psi_a k_1 \lambda}{\sinh k_1 a} \gamma_S$ proportional to $\gamma_S$:
Fully shielded case: Plots of the jump in the gradient of $\psi$, vs. $y$ for $\mu_0 = 1.4$ and selected small values of $\alpha$, showing the occurrence of current-density reversal for the two smallest values.
Current reversals cause \( \frac{1}{2} \) islands!

Ripple amplitude: \( \alpha = 0.003 \)
Current density exhibits sign reversal

2-region MRxMHD Hahm-Kulsrud model: mirror-image ripple top and bottom excites modulated current sheet at \( x = 0 \)

Larger ripple amplitude: \( \alpha = 0.005 \)
No sign reversal so half-islands disappear
Poloidal flux as a function of $x_0$ ($= x$ along $y$-axis), showing discontinuity in slope at $x = 0$ caused by current sheet.

Toroidal flux as a function of $x$ along $y$-axis, showing discontinuity at $x = 0$ caused by half-island.

Rotational transform ($1/q$) showing jump or large slope near $x_0 = 0$. 

$\alpha = 0.001$ 

(Dashed curves are for plane slab, $\alpha = 0$)
$\alpha = 0.005$
(Dashed curves are for plane slab, $\alpha = 0$)

Discontinuity in toroidal flux has gone as there are no half-islands above a threshold in $\alpha$ c. 0.0045

Much stronger jump in rotational transform
Full $t$-dependence: linear modes in slab

$B_1$ a superposition of “Beltrami waves” in plasma ($\mu > 0$) and vacuum ($\mu = 0$)

New MRxMHD: sound waves in plasma ($\rho_0 = \text{const} > 0$, $\tau > 0$)

Old MRxMHS+: $\lambda = \omega^2$ with $\rho_0 = \delta(x-a)$ ⇒ no sound waves

Hole et al, Nucl Fusion 47, 746 (2007), etc
Alexis Tuen’s MSc thesis 2016
First two eigenvalues, + incompressible approximation at very small $\lambda \equiv \omega^2$

- Growth rate zero if wall or $k \cdot B = 0$ is at interface
Detailed Conclusions

• Multi-region generalization of Taylor relaxation has been extended to a self-consistent dynamics through Hamilton’s Principle of Stationary Action.

• A rippled slab model has been used to illustrate the formation of a resonant current sheet as boundary ripple is switched on.

• For very small ripple amplitudes current reversal occurs in the current sheet and unperturbed sheared magnetic field exhibits topological change, with small half-islands, locking rotational transform to resonant value.

• For larger ripple amplitude the rotational transform jumps across the current sheet.
General Conclusion

- Action-based MRxMHD shows great promise
  - Very simple
  - Includes reconnection and flow in natural way
- $1^{st}$: check physical reasonability of predictions in simple models
- $2^{nd}$: Extend SPEC to flow; build new time-evolution and normal mode codes