Hall Effect on Equilibrium, Stability and Wave Spectrum of Magneto-Fluid Plasmas

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Chapter 1

Introduction

The processes of flow and the resultant flow-magnetic field couplings (induction and acceleration) are the most important characteristics of the collective behavior of plasmas. Dynamics and structures become far richer when an ambient flow exists in a plasma. Despite a long list of interesting subjects related to plasma flows, such as mixing and stretching effects, stabilization/destabilization of various waves, shocks, and discontinuities, the most fundamental part of the theories – equilibrium with flow, the linear stability of perturbations – still needs to be carefully examined [1, 2].

Magnetohydrodynamics (MHD) is widely used as the macroscopic theory of electrically conducting fluids, providing a theoretical framework for describing both laboratory and astrophysical plasmas. The MHD description is seemingly valid for a static and macroscopic system. In contrast to the name of magneto-hydrodynamics, however, the MHD can capture only a rather small aspect of plasma flows. In fact, most MHD studies of plasmas deal with magnetostatic configurations. This is not only because of a convenience but because powerful mathematical methods have been developed for flow-less MHD plasmas, for example, the Grad-Shafranov equation for equilibrium [3], the energy principle for linear stability theory [4] and so on [5, 6]. Conversely, one might say that there are many difficulties in investigating flowing plasmas. We will show some difficulties of MHD with flow in brief. Within scope of MHD theory, it is difficult to describe the complex flow patterns observed in real plasmas, such as the plasma flow that has a perpendicular component to the magnetic field. Actually, considering the equilibrium of flowing plasmas in two-dimensional geometry (like an axisymmetric tokamak device), the stream function of flow field must be a function of the magnetic flux function, i.e., the poloidal component of flow cannot cease to be parallel to the poloidal magnetic field in the axisymmetric torus geometry. It leads the fact that the partial differential equation describing the equilibrium has a singularity at the point where the poloidal flow velocity coincides with the poloidal Alfvén velocity, which is defined by the poloidal magnetic field. The equilibrium equation with compressible flow has another more pathological difficulty that the type of equation changes alternatively between elliptic and hyperbolic depending on the poloidal Alfvén Mach number [1, 2]. The existence of hyperbolic regimes implies that we cannot solve the equilibrium equation with a boundary condition because the hyperbolic partial differential equation requires a Cauchy date. It is also well known that, in the area of linear stability analysis of MHD, it is generally not possible to give necessary and sufficient condition for stability of flowing plasmas because the generator of flow dynamics becomes non-Hermitian (non-self-adjoint) operator, in which system the spectral decomposition is not assured. Unlike the static case, the spectral analysis (dispersion relation) of the generator to find mode frequencies of perturbations does not lead to a complete understanding of stability of flowing plasmas because the modes are not necessarily independent; they compose a nonorthogonal set of elements of Hilbert space. Interactions among the different modes may bring about a variety of transient phenomena [7, 8, 9, 10, 11]; the existence of an algebraic growing instability in a system where the entire frequency spectrum is real is an example of the pathology in the system. Furthermore, the MHD equation, which does not have an intrinsic scale length, is not capable of simulating the dynamics associated with the smaller scale than the system size. The small scale component may play an important role in various plasma phenomena, such as coronal heating [12, 13], magnetic reconnection [14]. There is a widespread realization that the evolution of macroscopic structures in the laboratory-created as well as natural plasmas must be aided by physical processes at microscopic scales associated with individual particle motion. This inference is physically sensible even though we may not find explicit expressions of these scales in the observable of a given structure. The theories behind the reconnection of flux tubes, flares or accretion discs are a few examples where attempts are made to understand the macro phenomena in terms of the invisible microdynamics. The macrodynamics of all these systems are governed by (almost) MHD, while the microdynamics are not the case.

In order to describe magneto-fluid plasma dynamics in more detail, one can employ the two-fluid effect. The two-fluid model of a plasma describes the coupling between the magnetic and the fluid aspects of the plasmas, and then helps us understand a variety of structures generated in plasmas [15]. Since the two-fluid model can capture the difference between the ion flow, which is approximately considered as plasma flow, and the electron flow, which moves almost parallel to the magnetic field, one can take into account more complicated magnetic and flow fields in more proper way than one-fluid model (MHD), for instance, the perpendicular flow field to the magnetic field. Neglecting the electron inertia, the two-fluid model may be written by a simple form, that is the Hall MHD. The Hall MHD is defined to be standard one-fluid MHD plus the Hall (current) term in the induction equation (Ohm's law), therefore the Hall term is the principal term distinguishing the two-fluid model from the one-fluid model. The Hall term, usually assumed to be small, is expressed by a high spatial derivative term (i.e., mathematically a singular perturbation of MHD) that introduces a short characteristic length scale (the ion skin depth) to the otherwise scale-less MHD. It becomes possible, thus, to have equilibria in which related physical quantities can vary on vastly different length scales [16]. Since the system degenerates into the standard scaleless MHD for negligible (or a specific kind of) flows, the two-fluid (Hall) effect strongly relates to the plasma flow. It may safely be said that the two-fluid (Hall) MHD model is more appropriate for dealing with plasma dynamics associated with the flow field and (or) the small scale behavior.

We will investigate the Hall effect, which is the principal two-fluid effect, on the equilibrium with plasma flow, the stability analysis of flowing plasmas and the wave spectrum. We start with comparison between the conventional

MHD and the two-fluid (Hall) MHD in Chap. 2. The MHD and the Hall MHD equations are derived from the two-fluid description of magneto-fluid plasmas, and the singular perturbation effect of the Hall term are pointed out in Sec. 2.1. The simplest and perhaps the most fundamental equilibrium state in a vortex dynamics system is introduced in Sec. 2.2. This state is defined by "Beltrami condition," an expression of the alignment of a vorticity with its flow [17]. The Beltrami condition gives an important class of equilibria, which is regarded as a relaxed state in relation to self-organizing theory [18, 19, 20]. The Beltrami condition on the two-fluid model, which can be cast in the coupled vortex dynamics form, leads the double Beltrami field that is expressed by a combination of two different Beltrami fields (eigenfunctions of the curl operator) [16, 21]. It shows directly that the double Beltrami field has two scale length, and then can deal with the different scale interaction, for example one is a large scale determined by the macroscopic structure, while the other is the intrinsic scale (the ion skin depth). The double Beltrami fields, tight coupling and well-defined combination of the magnetic and flow-velocity fields, span a far richer set of plasma conditions than the single Beltrami fields in standard MHD. This set of relaxed states, despite the simple mathematical structure, includes a variety of plasma states that could explain a host of interesting phenomena [22, 23, 24]. The essential new physics is due to the Hall term that relates the kinematic and the magnetic aspects of magneto-fluid plasmas.

In Chap. 3, we investigate the equilibrium of flowing plasmas. As mentioned above, the MHD equilibrium state with compressible flow is determined by the equation that is not always elliptic [1, 2] and for which the boundary value problem may not be appropriate. It is discussed in Sec. 3.1. Assuming incompressible flows (plasma flows can be regarded as incompressible in many interesting phenomena), the MHD equilibrium in axisymmetric two-dimensional geometry obeys an analog of the Grad-Shafranov equation [25]. The equation becomes elliptic partial differential equation, however, unlike the Grad-Shafranov equation for static equilibrium, it has a singularity when the poloidal Alfvén Mach number is unity. The appearance of the singularity is caused by the restriction that the poloidal flow cannot deviate from the magnetic surface (the poloidal magnetic field). On the other hand, for the two-fluid MHD, the equilibrium equation in axisymmetric geometry is cast in coupled elliptic Grad-Shafranov type equations without any singularity. This is because of a singular perturbation effect of the Hall term, which allows the poloidal flow to be perpendicular to the poloidal magnetic field and removes the MHD singularity. Therefore the coupled Grad-Shafranov equations for two-fluid MHD can be solved with proper boundary condition and we get the double Beltrami equilibrium in axisymmetric geometry in Sec. 3.2. In Sec. 3.3, by analyzing the double Beltrami field, we demonstrate that physical effects, which translate as a singular perturbation and a plasma flow, create a scale-hierarchy in the original system (with a single macroscopic scale). Perpendicular flow, which is allowed by the Hall term, is important again here to generate a small scale length.

We study the Lyapunov stability analysis of flowing plasmas in Chap. 4. As we mentioned before, unlike the static case, the standard spectral analysis cannot provide a complete understanding of stability in the non-Hermitian system of flowing plasmas. In Sec. 4.1, we show a general abstract theorem for the stability analysis of a special class of flows, which is called Beltrami flow. In connection with a variational principle characterizing the Beltrami flow, we have a constant of motion that bounds the energy of perturbations and leads a sufficient (Lyapunov) stability condition. This stability condition suppresses any instability including non-exponential (algebraic) growth due to non-Hermitian generator; it also prohibits nonlinear evolution. The key to prove is the coerciveness of the constant of motion in the topology of the energy norm. The theory is applied to Beltrami flows in MHD and the stability condition is obtained in Sec. 4.2. However, this method fails to analyze the stability of double Beltrami flows in two-fluid MHD because the singular perturbation term (mainly expressed by the Hall term) destroys the coerciveness of the constant of motion [26, 27]. The stability analysis of the double Beltrami flows requires a certain stronger (more coercive) constant, which corresponds to an enstrophy order constant. It is shown in Sec. 4.3 that, although the enstrophy is not generally a constant because of a vortex-stretching effect, an enstrophy order constant can be found for a special class of the double Beltrami flows and a Lyapunov function that bounds the energy of possible perturbations can be constructed.

In Chap. 5, we investigate the Hall effect on the Alfvén wave spectrum. In MHD, the Alfvén wave (the dominant low frequency mode of a magnetized plasma) displays a continuous spectrum associated with singular eigenfunctions [28, 29, 30, 31]. However, in a more realistic treatment of the plasma, some singular perturbation effects are expected to convert the continuous to the point spectrum. Such qualitative changes in spectrum are relevant to many plasma phenomena and lead interesting physical phenomena [32]. For example, adding the finite resistivity leads the tearing mode instability represented by an imaginary frequency point spectrum occurring through a singular perturbation of the edge of the Alfvén continuous spectrum [33]. Electron inertia or other kinetic effects are also well known to resolve the Alfvén singularity by leading a singular perturbation represented by a higher order derivative term to the mode equation [34]. Since the Hall term is added to MHD as a singular perturbation, it may be expected to remove the Alfvén singularity of MHD. It is shown that the coupling of the Hall current with the sound wave induces higher (fourth) order derivative in the Alfvén mode equation, and by resolving the singularity replaces the MHD continuum by a discrete spectrum. The mode structure resulting from the Hall resolution of the singularity is compared with the standard electron-inertia approach.

In Chap. 6, we summarize our results.

Chapter 2

Two-fluid model and Beltrami field

2.1 MHD and two-fluid (Hall) MHD

We start with the ideal two-fluid description of magneto-fluid plasmas and derive the Hall MHD and the conventional MHD. For magneto-fluid plasmas, the two-fluid description can be derived in a systematic mannar starting from the kinetic equations for ions and electrons, as done for instance by Braginskii [35]. The two-fluid model for the macroscopic dynamics of a plasma takes into account the difference between the electron and the ion flow velocities.

Denoting the electron and ion flow velocity by $\boldsymbol{V}_e$ and $\boldsymbol{V}_i$, the macroscopic evolution equations become

$$\partial_t \boldsymbol{V}_e + (\boldsymbol{V}_e \cdot \nabla)\boldsymbol{V}_e = \frac{-e}{m}(\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}) - \frac{1}{mn}\nabla p_e, \tag{2.1}$$

$$\partial_t \boldsymbol{V}_i + (\boldsymbol{V}_i \cdot \nabla)\boldsymbol{V}_i = \frac{e}{M}(\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}) - \frac{1}{Mn}\nabla p_i, \tag{2.2}$$

where $\boldsymbol{E}$ is the electric field, p_e and p_i are, respectively, the electron and the ion pressures, e is the elementary charge, n is the number density of both electrons and ions (we consider a quasi-neutral plasma with singly charged ions), m and M are the electron and the ion masses, respectively. In the electron equation, the inertial terms [left-hand side of (2.1)] can be safely neglected, because of their small mass. Therefore, (2.1) reduces to

$$\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B} + \frac{1}{en}\nabla p_e = 0. \tag{2.3}$$

When electron mass is neglected, the fluid velocity can be approximately expressed by the ion flow velocity, since $\boldsymbol{V} = (m\boldsymbol{V}_e + M\boldsymbol{V}_i)/(m+M) \simeq \boldsymbol{V}_i \quad (m \ll M)$.

We write $\boldsymbol{V}_e = \boldsymbol{V} - \boldsymbol{j}/(en) = \boldsymbol{V} - \nabla \times \boldsymbol{B}/(\mu_0 en)$ and $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi$, where $\boldsymbol{j}$ is the current density, μ_0 is the vacuum permeability, $\boldsymbol{A}$ and ϕ are the vector and scalar potential, respectively. Furthermore, choosing representative measures of the length L_0 and the magnetic field B_0, we introduce the following set of dimensionless variables,

$$
\begin{cases}
\boldsymbol{x} = L_0 \hat{\boldsymbol{x}}, \quad \boldsymbol{B} = B_0 \hat{\boldsymbol{B}} \\
t = (L_0/V_A)\hat{t}, \quad p = (B_0^2/\mu_0)\hat{p}, \quad \boldsymbol{V} = V_A \hat{\boldsymbol{V}}, \\
\boldsymbol{A} = (L_0 B_0)\hat{\boldsymbol{A}}, \quad \phi = (V_A L_0 B_0)\hat{\phi},
\end{cases}
\tag{2.4}
$$

where the Alfvén velocity is given by $V_A = B_0/\sqrt{\mu_0 M n}$ (we assume $\nabla \cdot \boldsymbol{V} = 0$ and n = constant, for simplicity). Then (2.3) and (2.2) transform to the dimensionless set of equations,

$$
\partial_{\hat{t}} \hat{\boldsymbol{A}} = \left(\hat{\boldsymbol{V}} - \varepsilon \hat{\nabla} \times \hat{\boldsymbol{B}}\right) \times \hat{\boldsymbol{B}} - \hat{\nabla}\left(\hat{\phi} - \varepsilon \hat{p}_e\right), \tag{2.5}
$$

$$
\begin{aligned}
\partial_{\hat{t}}(\varepsilon \hat{\boldsymbol{V}} + \hat{\boldsymbol{A}}) = \hat{\boldsymbol{V}} \times \left(\hat{\boldsymbol{B}} + \varepsilon \hat{\nabla} \times \hat{\boldsymbol{V}}\right) \\
- \hat{\nabla}\left(\hat{\phi} + \varepsilon \hat{V}^2/2 + \varepsilon \hat{p}_i\right),
\end{aligned}
\tag{2.6}
$$

where the scaling coefficient $\varepsilon = l_i/L_0$ is a measure of the ion skin depth,

$$
l_i = \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}} = \sqrt{\frac{M}{\mu_0 n e^2}}.
$$

In what follows, we will drop the hat ˆ on the normalized variables to simplify notation. Subtracting (2.5) from (2.6), and taking curl of (2.5), we get a set of Hall MHD equations,

$$
\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{2.7}
$$

$$
\partial_t \boldsymbol{B} = \nabla \times \left[(\boldsymbol{V} - \varepsilon \nabla \times \boldsymbol{B}) \times \boldsymbol{B}\right]. \tag{2.8}
$$

The only addition to the standard MHD is the Hall-term $\varepsilon(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ in (2.8). Therefore, setting $\varepsilon = 0$, we obtain the standard MHD equation. Mathematically ε is a singular perturbation parameter, since it multiplies the highest derivative term in (2.8). In some plasmas, ε may not be small, so that one can say that the Hall term represents a singular deformation of MHD. Generally speaking, Hall MHD theory is relevant to plasma dynamics occurring on length scales of the ion skin depth.

2.2 Beltrami condition

The "Beltrami condition," an expression of the alignment of a vorticity with its flow, describes the simplest and perhaps the most fundamental equilibrium state in a vortex dynamics system [17]. It is also believed that the resulting fields are self-organized in turbulent flows [20, 36].

A general solenoidal vector field, such as a magnetic field or an incompressible flow, can be decomposed into an orthogonal sum of Beltrami fields (eigenfunctions of the curl operator) [37]. Nonlinear dynamics of a plasma induces complex couplings among these Beltrami fields. In a one-fluid MHD plasma, however, the energy condensates into a single Beltrami magnetic field resulting in the self-organization of a force-free equilibrium, that is, the Taylor relaxed state (Sec. 2.2.1). By relating the velocity and the magnetic fields, the Hall term in the two-fluid model leads to a singular perturbation that enables the formation of an equilibrium given by a pair of two different Beltrami fields (Sec. 2.2.2).

We start with reviewing the prototype equation for vortex dynamics. Let $\boldsymbol{\omega}$ be a three dimensional vector field representing a certain vorticity in $\mathbb{R}^3$. We consider an incompressible flow $\boldsymbol{U}$ that transports $\boldsymbol{\omega}$. When the circulation associated with the vorticity is conserved everywhere, this $\boldsymbol{\omega}$ obeys the equation

$$\partial_t \boldsymbol{\omega} - \nabla \times (\boldsymbol{U} \times \boldsymbol{\omega}) = 0. \tag{2.9}$$

The general steady states of (2.9) are given by

$$\boldsymbol{U} \times \boldsymbol{\omega} = \nabla \varphi, \tag{2.10}$$

where φ is a certain scalar field, which physically corresponds to the energy density (pressure) in the original (de-curled) equation.

The "Beltrami condition," which demands alignment of vortices and flows, is expressed by

$$\boldsymbol{\omega} = \mu \boldsymbol{U}, \tag{2.11}$$

where μ is a certain scalar function. The Beltrami condition (2.11), thus, gives a special class of solution such that

$$\boldsymbol{U} \times \boldsymbol{\omega} = 0 = \nabla \varphi. \tag{2.12}$$

The former equality is the Beltrami condition, while the latter, demanding that the energy density is homogeneous, is a "generalized Bernoulli condition." The generalized Bernoulli condition demands that the constancy of the energy density refers to the direction perpendicular, as well as parallel, to the stream line $\boldsymbol{U}$. This is an essential difference from the conventional Bernoulli condition. It might appear that the Beltrami-Bernoulli states are very special and may be inaccessible. These conditions, however, follow from the concept of relaxed states. Indeed, the generalized Bernoulli condition describes homogeneous distributions of the energy density. Therefore, it is believed that the Beltrami fields are accessible and robust in the sense that they emerge as the nonlinear dynamics of vortices tends to self-organize the system through a weakly dissipative process [20, 36]. This is why the Beltrami fields are considered as relaxed states in the system.

We have to mention that the vortex governed by (2.9) has an important invariant, that is the "helicity." Under the boundary condition

$$\boldsymbol{n} \times (\boldsymbol{U} \times \boldsymbol{\omega}) = 0, \tag{2.13}$$

where $\boldsymbol{n}$ is the unit normal vector onto the boundary, the general helicity is defined as

$$H = \frac{1}{2} \int_\Omega (\mathrm{curl}^{-1} \boldsymbol{\omega}) \cdot \boldsymbol{\omega} \; \mathrm{d}^3 x, \tag{2.14}$$

where curl^{-1} is the inverse operator of the curl that is represented by the Biot-Savart integral. By this definition, we easily verify the conservation of H.

The simplest example of the vortex dynamics equation is that of the Euler equation of incompressible ideal flows. Let $\boldsymbol{U}$ be an incompressible flow that obeys

$$\partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\nabla p, \tag{2.15}$$

where p is the pressure. Taking the curl of (2.15), we obtain the evolution equation for the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{U}$, which reads as (2.9). It is well known that the kinetic helicity

$$\frac{1}{2} \int_\Omega \boldsymbol{U} \cdot \boldsymbol{\omega} \; \mathrm{d}^3 x, \tag{2.16}$$

is an invariants of the Euler equation.

The Beltrami flow is expressed by

$$\nabla \times \boldsymbol{U} = \mu \boldsymbol{U}. \tag{2.17}$$

We note that (2.17) is not Galilei transform invariant. We thus consider a bounded domain and impose a boundary condition to remove the freedom of the Galilei transformation (see *Remark* below). Taking the divergence of (2.17), we find that the scalar function μ must satisfy

$$\boldsymbol{U} \cdot \nabla\mu = 0, \tag{2.18}$$

demanding that μ must remain constant along each streamline of the flow $\boldsymbol{U}$. An analysis of the nonlinear system of elliptic-hyperbolic partial differential equations (2.17) - (2.18) involves extremely difficult mathematical issues. The characteristic curve of (2.18) is the streamline of the unknown flow $\boldsymbol{U}$, which can be chaotic (non-integrable) in general three-dimensional problems. Two special cases, however, can be studied rigorously [38]. One is the case where $\boldsymbol{U}$ has a coordinate that can be ignored (two-dimensional). Then, the streamline equation becomes integrable, and the system (2.17) - (2.18) reduces into a nonlinear elliptic equation. The other one is the case where μ is a constant number. It makes (2.18) trivial and the analysis reduces into a simple but nontrivial problem, i.e., the eigenvalue problem of the curl operator. Let us briefly review the mathematical theory of constant-μ (homogeneous) Beltrami fields.

Let Ω be a bounded three-dimensional domain with a smooth boundary Γ. We assume that Ω is multiply connected with cuts Σ_ℓ [$\ell = 1, \dots, m$ (the first Betti number)], i.e., $\Omega \setminus \cup(\Sigma_\ell)$ is simply connected.

Remark. In a multiply connected domain Ω ($\subset \mathbb{R}^3$), the curl operator has a point spectrum that covers the entire complex plane [37]. This is because of the existence of a nonzero harmonic field ($\nabla \times \boldsymbol{h} = 0$, $\nabla \cdot \boldsymbol{h} = 0$ in Ω, and $\boldsymbol{n} \cdot \boldsymbol{h} = 0$ on Γ), which plays the role of an inhomogeneous term in the eigenvalue problem

$$\begin{cases} \nabla \times \boldsymbol{U} = \mu \boldsymbol{U} & (\text{in } \Omega), \\ \boldsymbol{n} \cdot \boldsymbol{U} = 0 & (\text{on } \Gamma), \end{cases} \tag{2.19}$$

where $\boldsymbol{n}$ is the unit normal vector on Γ. We decompose the solenoidal field $\boldsymbol{U}$ into the harmonic component $\boldsymbol{h}$ and its orthogonal compliment $\boldsymbol{u}_\Sigma$. We can show that the latter component is a member of the Hilbert space

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{a} \in L^2(\Omega);\ \boldsymbol{n} \times \boldsymbol{a} = 0 \text{ on } \Gamma\}.$$

The eigenvalue problem now reads as

$$\nabla \times \boldsymbol{u}_\Sigma = \mu(\boldsymbol{u}_\Sigma + \boldsymbol{h}).$$

If we take $\boldsymbol{h} = 0$, we find a nontrivial solution only for $\mu_j \in \sigma_p$, where σ_p a countably infinite set of real numbers. The set σ_p constitutes the point spectrum of the self-adjoint curl operator that is defined in the Hilbert space $L^2_\Sigma(\Omega)$. For $\mu' \notin \sigma_p$, we must invoke $\boldsymbol{h} \neq 0$ and find a solution $\boldsymbol{u}_\Sigma = (\text{curl} - \mu')^{-1}\mu'\boldsymbol{h}$, where curl denotes the self-adjoint curl operator [37]. When the domain Ω is multiply connected, therefore, we can assume that the Beltrami parameters μ are arbitrary real (and even complex) numbers.

2.2.1 Single Beltrami field in MHD

We show the Beltrami condition in MHD, which is the set of equations (2.7) - (2.8) with $\varepsilon = 0$. Taking the curl of (2.7), we obtain a set of vortex equations;

$$\partial_t(\nabla \times \boldsymbol{V}) - \nabla \times [\boldsymbol{V} \times (\nabla \times \boldsymbol{V}) + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}] = 0, \tag{2.20}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = 0. \tag{2.21}$$

The Beltrami conditions for this system of vortex dynamics equations are summarized as

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}, \tag{2.22}$$

$$\boldsymbol{V} = \mu \boldsymbol{B}, \tag{2.23}$$

where λ and μ are assumed constants [17]. Since the field $\boldsymbol{B}$ (or $\boldsymbol{V}$) obeys "single-curl" equation (2.22), we call it "single" Beltrami field (in order to distinguish from "double" Beltrami field introduced in the following section). For

$\lambda \neq 0$ in (2.22), the Beltrami magnetic field $\boldsymbol{B}$ has a finite curl, and hence, the field lines are twisted. The current (proportional to $\nabla \times \boldsymbol{B}$), flowing parallel to the twisted field lines, creates what may be termed as "paramagnetic" structures. The magnetic configuration expressed by (2.22), for which the magnetic stress $\boldsymbol{j} \times \boldsymbol{B}$ vanishes, is aptly called "force-free" field. In order to characterize the stellar magnetic fields, the single Beltrami fields (2.22), especially flow-less $\mu = 0$ in (2.23), were intensively studied in 1950s [39, 40, 41]. Such twisted magnetic field lines appear commonly in many different plasma systems such as the magnetic ropes created in solar and geo-magnetic systems [42], and galactic jets [43]. Solar flares and coronal mass ejections (CMEs) creates magnetic clouds, causing interplanetary shocks, which are traveling shocks propagating out through the solar system. The magnetic field lines in the magnetic cloud were shown to have a helical geometry, which is considered as an approximately force-free (Beltrami) flux rope [44, 45]. On the other hand, some laboratory experiments have also shown that the "Taylor relaxed state" generated through turbulence is well described by the single Beltrami fields, the solutions of (2.22) [18, 19, 46].

2.2.2 Double Beltrami field in two-fluid MHD

We will see the Beltrami condition in two-fluid MHD, which is expressed by the set of equations (2.5) - (2.6) with $\varepsilon \neq 0$. The two-fluid MHD, a more adequate formulation of the plasma dynamics, allows a much wider class of special equilibrium solutions. The set of new solutions contains field configurations that can be qualitatively different from the force-free magnetic fields [16, 21].

Taking the curl of (2.5) and (2.6), we can cast them in a revealing symmetric vortex equation

$$\partial_t \boldsymbol{\omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\omega}_j) = 0 \quad (j = 1, 2), \tag{2.24}$$

in terms of a pair of generalized vorticities

$$\boldsymbol{\omega}_1 = \boldsymbol{B}, \quad \boldsymbol{\omega}_2 = \boldsymbol{B} + \varepsilon \nabla \times \boldsymbol{V},$$

and the effective flows

$$\boldsymbol{U}_1 = \boldsymbol{V} - \varepsilon \nabla \times \boldsymbol{B}, \quad \boldsymbol{U}_2 = \boldsymbol{V}.$$

The vortex equation (2.24) indicates a coupling between the magnetic field and the plasma flow. The “Beltrami condition” (2.11), which implies the alignment of the vorticities and the corresponding flows, gives the simplest and perhaps the most fundamental equilibrium solution to (2.24). Assuming that a and b

are constants, the Beltrami condition reads as a system of simultaneous linear equations in $\boldsymbol{B}$ and $\boldsymbol{V}$,

$$\boldsymbol{B} = a(\boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B}), \tag{2.25}$$

$$\boldsymbol{B} + \varepsilon\nabla \times \boldsymbol{V} = b\boldsymbol{V}. \tag{2.26}$$

These equations have a simple and significant connotation; the electron flow $(\boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B})$ parallels the magnetic field $\boldsymbol{B}$, while the ion flow $\boldsymbol{V}$ follows the "generalized magnetic field" $(\boldsymbol{B} + \varepsilon\nabla \times \boldsymbol{V})$. This generalized magnetic field contains the fluid vorticity induced by the ion inertia effect on a circulating flow.

Combining (2.25) and (2.26) yields a second order partial differential equation

$$\varepsilon^2\nabla \times (\nabla \times \boldsymbol{u}) - \varepsilon\left(b - \frac{1}{a}\right)\nabla \times \boldsymbol{u} + \left(1 - \frac{b}{a}\right)\boldsymbol{u} = 0, \tag{2.27}$$

where $\boldsymbol{u} = \boldsymbol{B}$ or $\boldsymbol{V}$. It is convenient to denote the curl derivative $\nabla\times$ by "curl" to use it as an operator. Let us rewrite (2.27) in the form

$$(\mathrm{curl} - \lambda_+)(\mathrm{curl} - \lambda_-)\boldsymbol{u} = 0, \tag{2.28}$$

where

$$\lambda_\pm = \frac{1}{2\varepsilon}\left[(b - a^{-1}) \pm \sqrt{(b + a^{-1})^2 - 4}\right]. \tag{2.29}$$

Since the operators $(\mathrm{curl}-\lambda_\pm)$ commute, the general solution to (2.28) is given by the linear combination of the two Beltrami fields. Let $\boldsymbol{G}_\pm$ be the Beltrami fields such that

$$\begin{cases} (\mathrm{curl}-\lambda_\pm)\boldsymbol{G}_\pm = 0 & (\text{in } \Omega), \\ \boldsymbol{n}\cdot\boldsymbol{G}_\pm = 0 & (\text{on } \Gamma), \end{cases}$$

where Ω ($\subset \mathbb{R}^3$) is a bounded domain with a smooth boundary Γ and $\boldsymbol{n}$ is the unit normal vector onto Γ. Then, for arbitrary constants $C_\pm$, the double Beltrami magnetic field

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-, \tag{2.30}$$

solves (2.28). The corresponding flow is given by

$$\boldsymbol{V} = \left(\varepsilon\lambda_+ + \frac{1}{a}\right) C_+\boldsymbol{G}_+ + \left(\varepsilon\lambda_- + \frac{1}{a}\right) C_-\boldsymbol{G}_-. \tag{2.31}$$

Therefore, the existence of a nontrivial solution to the double curl Beltrami equations (2.27) will be predicated on the existence of the appropriate pair of single Beltrami fields. In a multiply connected domain, the solution of the double curl Beltrami equation is given by a linear combination of two single Beltrami fields that exist for arbitrary Beltrami parameters $\lambda_\pm$ (or a and b), see *Remark* above.

The double Beltrami field is characterized by four parameters; two eigenvalues (two scale length) $\lambda_\pm$ and two amplitudes $C_\pm$. In relating these parameters with macroscopic constants of motion (energy and two helicities), an interesting set of algebraic relations is derived. Using these relations, we observe that, when macroscale of interacting loop structures decreases sharply on condition that the energy is larger than some critical value, a fold catastrophe appears and an equilibrium is lost. Applying this model to the solar corona, it is shown that magnetic field energy transfer to flow is possible and solar eruptive events (flares, prominences, CMEs - coronal mass ejections) may happen [23, 24].

As we have shown in (2.12), the Beltrami condition directly leads the generalized Bernoulli condition, which expresses the homogeneous energy density. From (2.5) and (2.6), we find φ_j, which corresponds to the energy density of each fluid, as

$$\varphi_1 = \phi - \varepsilon p_e, \tag{2.32}$$

$$\varphi_2 = \phi + \varepsilon V^2/2 + \varepsilon p_i. \tag{2.33}$$

The generalized Bernoulli condition $\varphi_j =$ constant, implying that the energy density of the field is fully relaxed, gives a simple relation among the flow velocity, potential, and the static pressure. Subtracting (2.32) from (2.33) under the Bernoulli condition, we obtain

$$\beta + V^2 = \text{constant}, \tag{2.34}$$

where β is a conventional beta ratio that is given by $\beta = 2(p_e + p_i) = 2p$ in the normalized unit. This relation shows that the double Beltrami equilibrium is no longer zero-beta, but it can confine an appreciable pressure when an appreciable flow (in the Alfvén unit) is driven.

Finally, we make a small catalog of the known relaxed state equilibria in magneto-fluid plasmas and also point out how one may arrive at them. In Fig. 2.1, we may see a hierarchy determined by the increasing complexity of the final state. In supplying a magnetic field, current, and flow to the plasma, the energy of the system rises successively with the harmonic, the single, and the double Beltrami fields. These "energy levels" are explained as follows. Suppose that a plasma is produced in an external magnetic field (harmonic field). In the absence of a drive, such a plasma will disappear and the system will relax into the pure harmonic magnetic field ($\nabla \times \boldsymbol{B} = 0$). When a drive in the form of a plasma current is added, it sustains the total helicity, and the plasma relaxes into the Taylor state, which corresponds to the single Beltrami magnetic field (2.22) without flow, $\mu = 0$ in (2.23). When a strong flow exists in addition to the current in a two component plasma, the self-organized (relaxed) state may become the double Beltrami field (2.30) - (2.31) that is qualitatively different from the Taylor relaxed state. The new states represent a "singular perturbation" to the MHD accessible states because the two-fluid effect induces a coupling among the flow, magnetic field, electric field, and the pressure. As the final state becomes more and more complex, greater and greater care is needed for its creation and maintenance. However, if all the requirements are met, the more complex states can display a tremendously variegated and rich structure in field variations.

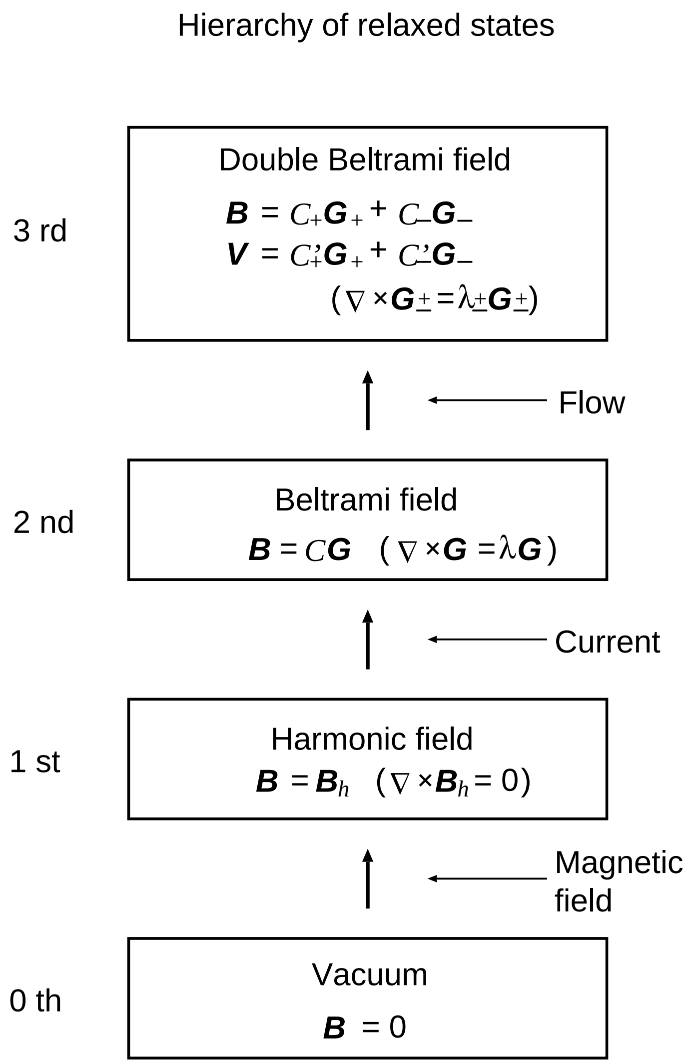

Figure 2.1: Hierarchy of relaxed states. The absolute minimum energy state is the vacuum. In supplying a magnetic field, current, and flow to the plasma, the energy of the system rises successively with the harmonic, the single, and the double Beltrami fields.

Chapter 3

Equilibrium of flowing plasmas

3.1 Characteristics of ideal MHD

Recently, in experimental and theoretical research of nuclear fusions, plasma flows in toroidal plasmas have attracted much attention. Theoretical studies of flowing plasmas begin in the mid-1950s, but most investigations of the equilibrium with plasma flow have done since the early 1970s ([1, 2, 25] and the references therein). Toroidal flows can be driven by Neutral Beam Injection (NBI) [47] and poloidal flows by radial electric fields [48]. It has been reported that flow shear can reduce macro- and microscopic instabilities of plasmas [49, 50].

Considering an axisymmetric geometry, the equilibrium state with plasma flow obeys a partial differential equation like the Grad-Shafranov equation for static equilibrium [3]. Unlike the case in static equilibria, however, the above-mentioned partial differential equation is not always elliptic: there are three critical values of the poloidal flow at which the type of this equation changes, i.e., it becomes alternatively elliptic and hyperbolic [1, 2]. The existence of hyperbolic regimes may be dangerous for plasma confinement because they are associated with shock waves which can cause equilibrium degradation. In this respect incompressible flows are of particular interest because, as is well known from gas dynamics, it is the compressibility that can give rise to shock waves; thus for incompressible flows the equilibrium equation becomes elliptic except a singular point where the poloidal Alfvén Mach number is unity [25].

Considering the Hall MHD, the singularity can be removed by the singular perturbation effect of Hall current. In one-fluid MHD the poloidal flow must be parallel to the poloidal magnetic field in the axisymmetric geometry (the stream function Φ must be a function of magnetic flux function Ψ). In Hall MHD, however, the poloidal flow may cease to be parallel to the poloidal magnetic field and the equilibrium state is governed by a coupled two elliptic partial differential equations for the magnetic flux function Ψ and the stream function

Φ, respectively. These equations are always elliptic without any singularity, then we can solve them under appropriate boundary condition.

We begin with the calculation of the characteristics of the ideal MHD system with compressible flow for general understanding. The system of equations may be written in the form

$$\partial_t n + \nabla \cdot (n\boldsymbol{V}) = 0, \tag{3.1}$$

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\,\boldsymbol{V} - n^{-1}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + n^{-1}\nabla p = 0, \tag{3.2}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = 0, \tag{3.3}$$

$$\partial_t p + \boldsymbol{V} \cdot \nabla p + nC_s^2 \nabla \cdot \boldsymbol{V} = 0, \tag{3.4}$$

where n is the plasma density and $C_s = \sqrt{\gamma p/n}$ (γ is the adiabatic constant) is the sound speed. The limit $\gamma \to \infty$ leads incompressible MHD.

A study of the property of the equations (3.1) - (3.4) may well start with the introduction of the characteristic surfaces of the system [51]. Let $\boldsymbol{U}$ represent the eight-dimensional vector $(n, \boldsymbol{V}, \boldsymbol{B}, p)$ and the differential equations may be written in the form

$$(A_t\partial_t + A_x\partial_x + A_y\partial_y + A_z\partial_z)\,\boldsymbol{U} = 0, \tag{3.5}$$

where the 8×8 matrices A_t, A_x, A_y and A_z are functions of $\boldsymbol{U}$ alone. With $\varphi(\boldsymbol{x}, t)$ denoting the eikonal function, (3.5) is rearranged to the linear algebraic system

$$(\omega A_t + k_x A_x + k_y A_y + k_z A_z)\,\frac{\partial \boldsymbol{U}}{\partial \varphi} = 0, \tag{3.6}$$

where $\omega = \partial_t \varphi$ and $\boldsymbol{k} = \nabla \varphi$.

The surfaces for which (3.6) has solutions are called characteristic surfaces, and for a surface to be characteristic

$$D(\omega, \boldsymbol{k}) \equiv \det |\omega A_t + k_x A_x + k_y A_y + k_z A_z| = 0. \tag{3.7}$$

The equation of the characteristics can be calculated in the following form

$$\begin{aligned} D &= \bar{\omega}^2 \left[\bar{\omega}^2 - (\boldsymbol{V}_A \cdot \boldsymbol{k})^2\right] \left[\bar{\omega}^4 - \left(C_s^2 + V_A^2\right) k^2 \bar{\omega}^2 + C_s^2 \left(\boldsymbol{V}_A \cdot \boldsymbol{k}\right)^2 k^2\right] \\ &= 0, \end{aligned} \tag{3.8}$$

where $\bar{\omega} = \omega + \boldsymbol{V} \cdot \boldsymbol{k}$ and $\boldsymbol{V}_A = \boldsymbol{B}/\sqrt{n}$. Comparing the linearized wave, the first and second terms of (3.8) corresponds to the entropy and Alfvén waves respectively, and the third one yields the fast and slow waves.

The characteristics of MHD steady state can be obtained by setting $\partial_t \varphi = \omega = 0$ in (3.8);

$$\begin{aligned} &(\boldsymbol{V} \cdot \boldsymbol{k})^2 [(\boldsymbol{V} - \boldsymbol{V}_A) \cdot \boldsymbol{k}][(\boldsymbol{V} + \boldsymbol{V}_A) \cdot \boldsymbol{k}] \\ &\quad \times \left[(\boldsymbol{V} \cdot \boldsymbol{k})^4 - (C_s^2 + V_A^2) k^2 (\boldsymbol{V} \cdot \boldsymbol{k}) + C_s^2 k^2 (\boldsymbol{V}_A \cdot \boldsymbol{k})^2\right] = 0. \end{aligned} \tag{3.9}$$

The first three terms in the left-hand side of (3.9) show hyperbolic parts of the equilibrium that require Cauchy data. The last term includes both hyperbolic and elliptic parts in complicated way. Assuming no-flow equilibrium ($\boldsymbol{V} = 0$), the last term can be rearranged to $k^2(\boldsymbol{V_A} \cdot \boldsymbol{k})^2$. Solutions of $(\boldsymbol{V_A} \cdot \boldsymbol{k})^2 = 0$ correspond to two hyperbolic parts, and $k^2 = 0$ correspond to two elliptic parts of the no-flow equilibrium equation, which require a boundary condition. In two-dimensional system, we can integrate the two hyperbolic parts by giving proper Cauchy data and get an elliptic partial differential equation called Grad-Shafranov equation that can be solved under an appropriate boundary condition [3]. However considering the steady flow, the situation is not so simple.

We will analyze (3.9) when the flow $\boldsymbol{V}$ is parallel to the magnetic field, i.e., $\boldsymbol{V_A}$. Let θ be an angle between $\boldsymbol{V}$ ($\boldsymbol{V_A}$) and $\boldsymbol{k}$. The last term of (3.9) is written as

$$k^4 \cos^2\theta \left(V^4 \cos^2\theta - (C_s^2 + V_A^2)V^2 + C_s^2 V_A^2\right) = 0. \tag{3.10}$$

When θ solves (3.10), the solution corresponds to hyperbolic part of the system, since it means nonzero $\boldsymbol{k}$ exists. We can easily find that two components of (3.10) are hyperbolic, because $\cos^2\theta = 0$ solve the equation. Then the problem is whether

$$V^4 \cos^2\theta - (C_s^2 + V_A^2)V^2 + C_s^2 V_A^2 = 0$$

can be solved for θ. The solvable condition can be calculated as

$$0 < V^2(V_A^2 + C_s^2) - V_A^2 C_s^2 < V^4. \tag{3.11}$$

In other words, when the equilibrium has a flow that does not satisfy (3.11), the system includes the elliptic parts. Namely, the flowing plasma equilibrium changes the type of equation depending on the flow velocity $\boldsymbol{V}$. We summarize the result in Fig. 3.1, where $\beta^* = C_s^2/V_A^2$ and $V_c^2 = V_A^2 C_s^2/(V_A^2 + C_s^2)$.

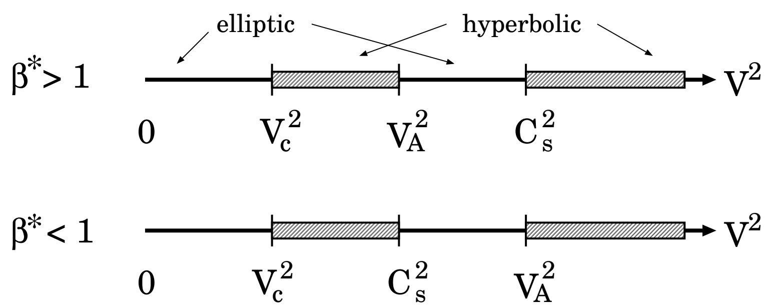

Figure 3.1: Elliptic and hyperbolic regions for parallel flow MHD.

When the system does not have any elliptic part, solving the equilibrium equation under the boundary condition is an ill-posed problem, since hyperbolic part require a Cauchy date instead of the boundary condition.

The most interesting steady state for fusion plasmas is an equilibrium in toroidal geometry, where we can assume homogeneity in toroidal direction. We end this section with analyzing this case. For this purpose we decompose the vector field into poloidal (represented by subscript p) and toroidal (represented by subscript t) component;

$$\boldsymbol{V}_A = \boldsymbol{V}_{Ap} + \boldsymbol{V}_{At}, \tag{3.12}$$

$$\boldsymbol{V} = \boldsymbol{V}_p + \boldsymbol{V}_t, \tag{3.13}$$

$$\boldsymbol{k} = \boldsymbol{k}_p, \tag{3.14}$$

where $\boldsymbol{k}_t = 0$ because of toroidal symmetry (the derivative with respect to the toroidal direction is zero). Furthermore the induction equation (3.3) requires that $\boldsymbol{V}_p$ must be align to $\boldsymbol{V}_{Ap}$, (which is equivalent to the fact that the stream function becomes a function of the magnetic flux surface). Introducing θ as an angle between $\boldsymbol{V}_p$ ($\boldsymbol{V}_{Ap}$) and $\boldsymbol{k}_p$, it is shown that the system does not have any elliptic part when θ solves

$$V_p^4 \cos^2\theta - (C_s^2 + V_A^2)V_p^2 + C_s^2 V_{Ap}^2 = 0.$$

The solvable condition is calculated as

$$0 < V_p^2(V_A^2 + C_s^2) - V_{Ap}^2 C_s^2 < V_p^4. \tag{3.15}$$

Then, we can summarize the result in the following way,

$$\begin{cases} 0 < V_p^2 < V_c^2 \quad \text{or} \quad V_s^2 < V_p^2 < V_f^2; & \text{elliptic,} \\ \\ V_c^2 < V_p^2 < V_s^2 \quad \text{or} \quad V_f < V_p^2; & \text{no elliptic,} \end{cases} \tag{3.16}$$

where

$$V_c^2 = \frac{C_s^2 V_{Ap}^2}{C_s^2 + V_{Ap}^2}, \tag{3.17}$$

$$V_{f,s}^2 = \frac{1}{2}\left[C_s^2 + V_A^2 \pm \sqrt{(C_s^2 + V_A^2)^2 - 4C_s^2 V_{Ap}^2}\right], \tag{3.18}$$

which is also shown in Fig. 3.2.

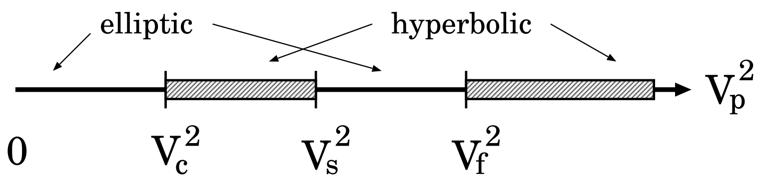

Figure 3.2: Elliptic and hyperbolic regions for flowing plasmas in toroidal geometry.

3.2 Grad-Shafranov equation of flowing plasma

3.2.1 MHD equilibrium with incompressible flow

The equilibrium equation for axisymmetric states can be casted in a second order partial differential equation, so called Grad-Shafranov equation. However, as mentioned in the previous section, the equation of the equilibrium with compressible flow changes the type of equation between elliptic and hyperbolic depending on the flow velocity. If we assume incompressible flow ($\nabla \cdot \boldsymbol{V} = 0$), the equation become a elliptic partial differential equation with singular point where poloidal Alfvén Mach number is unity.

We consider axisymmetric two-dimensional equilibria. Following the basic idea of formulating the Grad-Shafranov equation, we use the Clebsch representations of divergence-free axisymmetric ($\partial_\theta = 0$) vector function in cylindrical (r, θ, z) coordinates [3]. We write $\boldsymbol{B}$ and $\boldsymbol{V}$ in a contravariant-covariant combination form

$$\boldsymbol{B} = \nabla \Psi(r,z) \times \nabla \theta + r B_\theta(r,z)\, \nabla \theta, \tag{3.19}$$

$$\boldsymbol{V} = \nabla \Phi(r,z) \times \nabla \theta + r V_\theta(r,z)\, \nabla \theta, \tag{3.20}$$

where Ψ (or Φ) is the flux function (or the stream function) and B_θ (or V_θ) is the azimuthal magnetic (or velocity) field. All of them are function of only r and z (not θ). Substituting (3.19) - (3.20) into the following MHD equilibrium equations;

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - (\nabla \times \boldsymbol{V}) \times \boldsymbol{V} = \nabla P, \tag{3.21}$$

$$\boldsymbol{V} \times \boldsymbol{B} = \nabla \phi, \tag{3.22}$$

where $P = p + V^2/2$ and ϕ is the scalar potential, we can easily verify that $\Phi = \Phi(\Psi)$, $rB_\theta = rB_\theta(\Psi)$, $rV_\theta = rV_\theta(\Psi)$, $P = P(\Psi)$ and $\phi = \phi(\Psi)$, and obtain

$$-r^2 \nabla \cdot \left[\left(1 - \Phi'^2\right) \frac{\nabla \Psi}{r^2} \right] - \left(\frac{\Phi'^2}{2} \right)' |\nabla \Psi|^2 = F'(\Psi) + r^2 P'(\Psi), \tag{3.23}$$

or

$$-\left(1 - \Phi'^2\right) \mathcal{L}\Psi + \left(\frac{\Phi'^2}{2} \right)' |\nabla \Psi|^2 = F'(\Psi) + r^2 P'(\Psi), \tag{3.24}$$

where $F = (1/2)[(rB_\theta)^2 - (rV_\theta)^2]$, the prime indicates the derivative with respect to Ψ and $\mathcal{L}$ is a Grad-Shafranov operator

$$\mathcal{L} = r^2 \nabla \cdot \left(\frac{\nabla}{r^2} \right) = r\partial_r \frac{1}{r} \partial_r + \partial_z^2. \tag{3.25}$$

Note that the condition $\Phi = \Phi(\Psi)$ means that the poloidal (r-z) components of $\boldsymbol{V}$ and $\boldsymbol{B}$ are parallel and Φ' corresponds to the poloidal Alfvén Mach number. Taking $\Phi = 0$ and $V_\theta = 0$ (no-flow), (3.23) or (3.24) reduces into the well-known Grad-Shafranov equation [3]. Since $\mathcal{L}$ is an elliptic operator, the equation for an axisymmetric equilibrium with incompressible flow (3.23) or (3.24) is elliptic, however there is a singularity at $\Phi'^2 = 1$. The restriction $\Phi = \Phi(\Psi)$, limiting the perpendicular components of the flow, does not allow intersections of characteristics and causes the singularity generation.

3.2.2 Grad-Shafranov equation of two-fluid MHD

When we consider a steady state of the two-fluid (Hall) MHD in axisymmetric geometry, unlike one-fluid MHD, the stream function Φ can deviate from a function of the magnetic flux function Ψ because of the Hall effect, and it causes removing the singularity. The equilibrium equation for the two-fluid MHD becomes a set of coupled two elliptic partial differential equations for Ψ and Φ, which can be solved with a proper boundary condition. We will give an equilibrium solution of the double Beltrami field that is considered as a relaxed state of the two-fluid MHD (see Sec. 2.2.2).

From (2.5) - (2.6), the equilibrium state of the two-fluid MHD is governed by the following set of equations;

$$(\boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla\left(\varepsilon p_e - \phi\right) = 0, \tag{3.26}$$

$$\boldsymbol{V} \times (\boldsymbol{B} + \varepsilon\nabla \times \boldsymbol{V}) - \nabla\left(\varepsilon V^2/2 + \varepsilon p_i + \phi\right) = 0. \tag{3.27}$$

Substituting the Clebsch representation (3.19) - (3.20) into (3.26) - (3.27), we obtain

$$[\nabla(\Phi - \varepsilon r B_\theta) \times \nabla\Psi \cdot \nabla\theta]\, \nabla\theta + \frac{1}{r^2}(r V_\theta + \varepsilon \mathcal{L}\Psi)\, \nabla\Psi$$
$$- \frac{1}{r^2} r B_\theta \nabla(\Phi - \varepsilon r B_\theta) + \nabla(\varepsilon p_e - \phi) = 0, \qquad (3.28)$$
$$[\nabla\Phi \times \nabla(\Psi + \varepsilon r V_\theta) \cdot \nabla\theta]\, \nabla\theta + \frac{1}{r^2} r V_\theta \nabla(\Psi + \varepsilon r V_\theta)$$
$$- \frac{1}{r^2}(r B_\theta - \varepsilon \mathcal{L}\Phi)\, \nabla\Phi - \nabla\left(\varepsilon V^2/2 + \varepsilon p_i + \phi\right) = 0. \qquad (3.29)$$

From the component of (3.28) and (3.29) along $\nabla\theta$, we get

$$\Phi - \varepsilon r B_\theta = f_1(\Psi), \quad \text{or} \quad r B_\theta = (\Phi - f_1)/\varepsilon, \qquad (3.30)$$

$$\Psi + \varepsilon r V_\theta = g_1(\Phi), \quad \text{or} \quad r V_\theta = (g_1 - \Psi)/\varepsilon, \qquad (3.31)$$

where f_1 and g_1 are, respectively, arbitrary functions of Ψ and Φ. Plugging them in (3.28) and (3.29), we also get

$$\varepsilon p_e - \phi = f_2(\Psi), \qquad (3.32)$$

$$\varepsilon V^2/2 + \varepsilon p_i + \phi = g_2(\Phi), \qquad (3.33)$$

where f_2 and g_2 are also arbitrary functions of Ψ and Φ respectively, and then we have

$$P(\Psi, \Phi) = p + V^2/2 = (f_2 + g_2)/\varepsilon. \qquad (3.34)$$

Substituting (3.30) and (3.32) into (3.28), and (3.31) and (3.33) into (3.29), we obtain the following coupled Grad-Shafranov equations

$$-\varepsilon\mathcal{L}\Psi = -rB_\theta f_1' + rV_\theta + r^2 f_2', \tag{3.35}$$

$$-\varepsilon\mathcal{L}\Phi = -rB_\theta + rV_\theta g_1' - r^2 g_2', \tag{3.36}$$

where $f_i' = df_i/d\Psi$ and $g_i' = dg_i/d\Phi$. Defining

$$F(\Psi,\Phi) = \frac{1}{2}\left[(rB_\theta)^2 - (rV_\theta)^2\right], \tag{3.37}$$

the coupled Grad-Shafranov equations (3.35) and (3.36) can be written as

$$-\mathcal{L}\Psi = \partial_\Psi F(\Psi,\Phi) + r^2\partial_\Psi P(\Psi,\Phi), \tag{3.38}$$

$$-\mathcal{L}\Phi = -\partial_\Phi F(\Psi,\Phi) - r^2\partial_\Phi P(\Psi,\Phi). \tag{3.39}$$

Since the coupled Grad-Shafranov equations (3.35) - (3.36), or (3.38) - (3.39), are always elliptic partial differential equations and there is no singularity, unlike (3.23) or (3.24) for one fluid MHD equilibrium, we can solve them under a proper boundary condition. In the limit of $\varepsilon \to 0$, Φ must be a function of Ψ, i.e., $\Phi \to f_1(\Psi)$ in (3.30), and then the singularity may appear again in the equilibrium equation.

We end this section by giving the double Beltrami equilibrium in axisymmetric system. Using the Clebsch representation (3.19) - (3.20), the double

Beltrami condition (2.25) - (2.26) and the Bernoulli condition (2.34) lead the following relation

$$\begin{cases} \varepsilon r B_\theta = -a^{-1}\Psi + \Phi + C_1, \\ \varepsilon r V_\theta = b\Phi - \Psi + C_2, \\ P = \text{constant}, \end{cases} \tag{3.40}$$

where C_1 and C_2 are constants to be determined by flux (or boundary) conditions. The Beltrami condition demands linear functions $f_1(\Psi) = a^{-1}\Psi - C_1$ and $g_1(\Phi) = b\Phi + C_2$. Then, the coupled Grad-Shafranov equations are obtained as

$$-\varepsilon^2 \mathcal{L} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = \begin{pmatrix} a^{-2} - 1 & b - a^{-1} \\ a^{-1} - b & b^2 - 1 \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} - \begin{pmatrix} a^{-1}C_1 - C_2 \\ C_1 - bC_2 \end{pmatrix}. \tag{3.41}$$

We can solve (3.41) under a boundary condition $\Psi = \Phi = 0$ at the boundary. Figure 3.3 shows an example of the double Beltrami equilibrium where we set $\varepsilon = 1$, $a = 0.5$ and $b = 0.1$ in the axisymmetric system, in which we have an internal coil at $r = 4.5$ like an internal-ring devices Proto-RT [52] or Mini-RT [53] at the University of Tokyo. The plasma pressure (β) is determined by the Bernoulli condition (2.34). Since the pressure is not a function of the magnetic surface Ψ (or stream function Φ), the pressure may have a jump at the boundary.

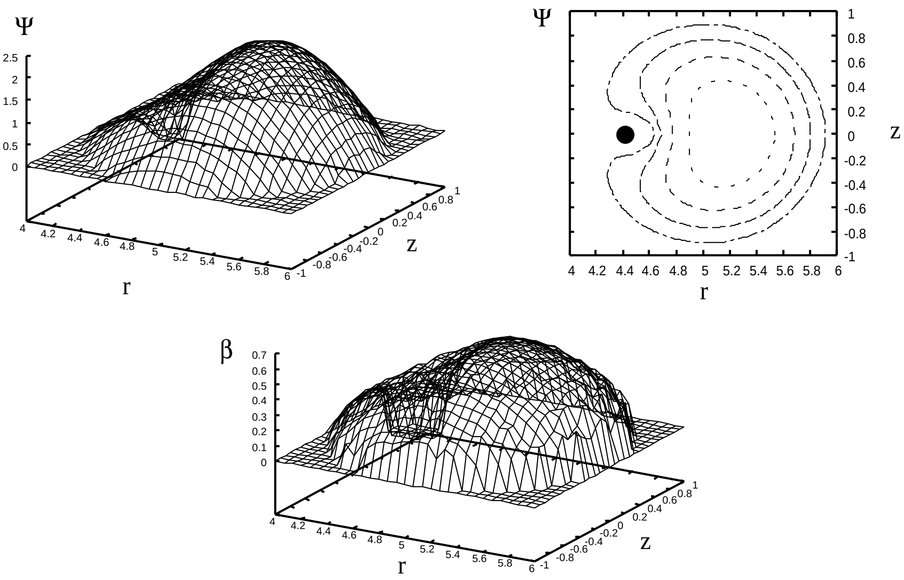

Figure 3.3: Axisymmetric equilibrium of double Beltrami field. Magnetic flux function Ψ and plasma pressure β.

3.2.3 Two-fluid MHD equilibrium with pure toroidal flow

In order to avoid the pressure jump at the boundary, we consider the non-constant density n keeping the assumption of flow incompressibility. The equilibrium equation is written as follows;

$$(n\boldsymbol{V} - \varepsilon\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \varepsilon\nabla p_e - n\nabla\phi = 0, \tag{3.42}$$

$$n\boldsymbol{V}\times(\boldsymbol{B}+\varepsilon\nabla\times\boldsymbol{V}) - \varepsilon n\nabla\frac{V^2}{2} - \varepsilon\nabla p_i - n\nabla\phi = 0, \tag{3.43}$$

$$\boldsymbol{V}\cdot\nabla n = 0. \tag{3.44}$$

We consider the simple but rather representative case of pure toroidal flow, i.e., $\Phi = 0$, (the electrons still have both toroidal and poloidal velocity components). With axisymmetric assumption, (3.44) is satisfied automatically. Substituting the Clebsch presentation (3.19) and $\boldsymbol{V} = rV_\theta\nabla\theta$ into (3.42) - (3.43), we obtain

$$\frac{\nabla\Psi}{r^2}\left(nrV_\theta + \varepsilon\mathcal{L}\Psi + \varepsilon II'\right) + \varepsilon\nabla p_e - n\nabla\phi = 0, \tag{3.45}$$

$$\frac{1}{r^2}nrV_\theta\nabla\left(\Psi + \varepsilon rV_\theta\right) - \varepsilon n\nabla\frac{V_\theta^2}{2} - \varepsilon\nabla p_i - n\nabla\phi = 0, \tag{3.46}$$

where $I = I(\Psi) = rB_\theta$. Subtracting (3.45) from (3.46), we get

$$-\frac{\nabla\Psi}{r^2}\left(\mathcal{L}\Psi + II'\right) + nV_\theta^2\nabla\ln r - \nabla p = 0, \tag{3.47}$$

where we use the relation

$$\frac{n}{r^2}rV_\theta\nabla rV_\theta - \frac{n}{2}\nabla\left(V_\theta\right)^2 = nV_\theta^2\nabla\ln r.$$

The component of (3.47) along $\boldsymbol{B}$ is put in the form

$$\boldsymbol{B}\cdot\left[nV_\theta^2\nabla\ln r - \nabla p\right] = 0. \tag{3.48}$$

When we can assume $nV_\theta^2 = h(r)J(\Psi)$, (3.48) is integrated to yield an expression for the pressure, i.e.,

$$p = P_s(\Psi) + H(r)J(\Psi), \tag{3.49}$$

where $P_s(\Psi)$ is the static part of the pressure, and $H(r)$ is defined as follows;

$$h(r)\nabla \ln r = \nabla H(r) \qquad (dH/dr = h/r)\,.$$

The pressure of (3.49) can be set 0 at the boundary as long as $P_s(\Psi) = J(\Psi) = 0$ at the boundary. With the aid of (3.49), (3.47) leads the Grad-Shafranov equation for the equilibrium with pure toroidal flow,

$$-\mathcal{L}\Psi = II' + r^2 P_s' + r^2 H(r) J'. \tag{3.50}$$

Next we will study what condition is needed to get the double Beltrami field with pure toroidal flow. The double Beltrami field is considered as a relaxed state based on constant density n_0 and may be expressed by

$$\begin{cases} \boldsymbol{B} = a\left(n_0 \boldsymbol{V} - \varepsilon \nabla \times \boldsymbol{B}\right) \\ \boldsymbol{B} + \varepsilon \nabla \times \boldsymbol{V} = b n_0 V, \end{cases} \tag{3.51}$$

and the Bernoulli condition

$$p + n_0 \frac{V^2}{2} = P_0, \tag{3.52}$$

where P_0 is a constant; see (2.25) - (2.26) and (2.34). Substituting the Clebsch presentation (3.19) and $\boldsymbol{V} = rV_\theta \nabla\theta$ into (3.51), we obtain

$$\begin{cases} \varepsilon r B_\theta = -a^{-1}\Psi + C_1, \\ \varepsilon r V_\theta = -\Psi + C_2, \end{cases} \tag{3.53}$$

and

$$rB_\theta = bn_0 rV_\theta, \tag{3.54}$$

which require $n_0 = 1/(ab)$ and $C_1 = bn_0C_2$. Then, (3.51) leads the equation for the double Beltrami field with pure toroidal flow as

$$-\varepsilon^2 \mathcal{L}\Psi = n_0 \left(b^2 n_0 - 1\right) \left(\Psi - C_2\right). \tag{3.55}$$

For the double Beltrami field, therefore, we have to set

$$\begin{cases} I(\Psi) = -bn_0 \left(\Psi - C_2\right)/\varepsilon, \quad J(\Psi) = n_0 \left(\Psi - C_2\right)^2/\varepsilon^2, \\ h(r) = 1/r^2, \quad \text{hence} \quad H(r) = -1/2r^2, \\ P_s(\Psi) = P_0, \end{cases} \tag{3.56}$$

in (3.50).

We will deviate P_s and n from P_0 and n_0 in oder to get a steady state such that the center of the plasma is expressed by the double Beltrami field (relaxed state) and the pressure connects to the boundary ($p = 0$) smoothly

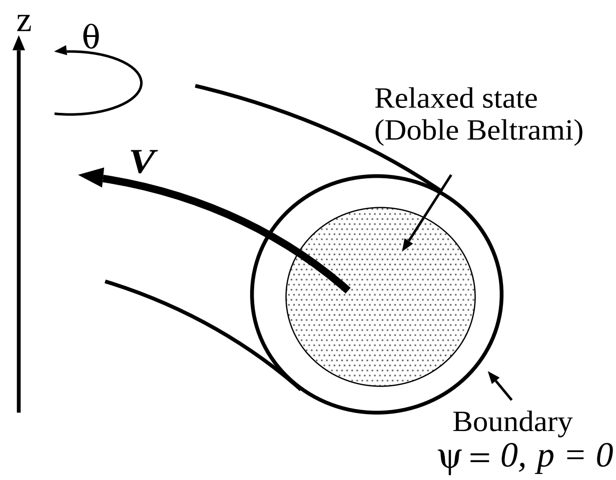

Figure 3.4: Schematic configuration of relaxed state with pure toroidal flow.

(without jump), like Fig. 3.4. For this purpose, P_s and n should be zero at the boundary $\Psi = 0$, and close to the constants, P_0 and n_0 respectively, in the center of the plasma $\Psi > 0$. As an example, setting n is a function of Ψ, we choose

$$\begin{cases} P_s(\Psi) = P_0 \left[1 - \operatorname{sech}(\alpha_p \Psi)\right], \\ n(\Psi) = n_0 \left[1 - \operatorname{sech}(\alpha_n \Psi)\right], \end{cases} \tag{3.57}$$

where α_p and α_n are constants. Replacing P_0 and n_0 in (3.56) to P_s and n of (3.57), the Grad-Shafranov equation is put in the form

$$-\varepsilon^2 \mathcal{L}\Psi = n\left(b^2 n - 1\right)(\Psi - C_2) + n'\left(b^2 n - 1/2\right)(\Psi - C_2)^2 + \varepsilon^2 r^2 P', \tag{3.58}$$

and the pressure is calculated by

$$p = P(\Psi) - \frac{n(\Psi)}{2\varepsilon^2} \frac{(\Psi - C_2)^2}{r^2}. \tag{3.59}$$

A typical solution of (3.58) - (3.59) is shown in Fig. 3.5.

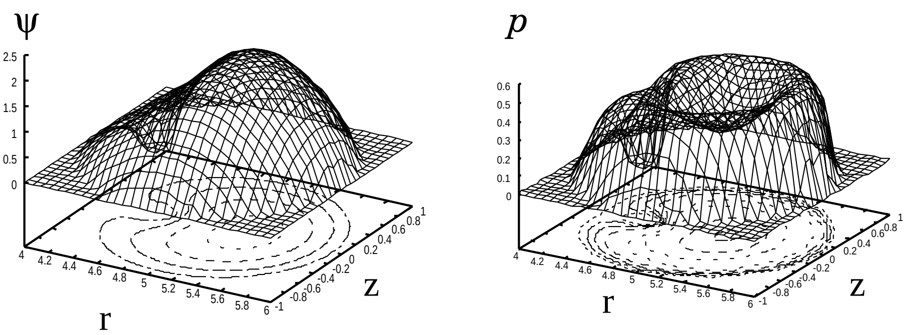

Figure 3.5: Double Beltrami field with pure toroidal flow. Magnetic flux function Ψ and plasma pressure p.

3.3 Scale hierarchy created in plasma flow

We show another important effect of singular perturbation in flowing plasmas, that is a creation of small scale length. Generally the physics controlling the microscopic scale appears as a singular perturbation to the macroscopic equations of motion, i.e, it enters through a term containing higher order derivatives multiplying a small coefficient (ε). We examine a particular solution of the two-fluid model (Hall MHD) that is the double Beltrami field (2.30) - (2.31). Since the double Beltrami field is expressed by two different Beltrami fields, it is decomposed as a sum of the universality class and the singular part, i.e., one of them approaches macroscopic solution of MHD and the other one is singular in the limit of $\varepsilon \to 0$. Within the framework of this model we will compare and contrast the virtues of two contending mechanisms for the creation of short-scale fields. Both these models, the relative recent double Beltrami relaxation model and the well-known current sheet model of Parker [12], have been invoked, for instance, to explain the heating of the solar corona. The former relies on the dissipation of the short scale during the relaxation process. It describes a "pattern" generated by the cooperation of the nonlinearity (convective type) and the dispersion (singular perturbation due to the Hall effect).

3.3.1 Small scale creation in two-fluid plasmas

The parameters $\lambda_\pm$ in (2.29), being the eigenvalues of the curl operator, characterize the reciprocal of the length scales on which $\boldsymbol{G}_\pm$ change significantly. We will assume that ε is much less than unity in order to study how small scale is created by the Hall term (singular perturbation). To view the solution and the associated scale lengths as explicit functions of the small parameter ε, let $|\lambda_-| = O(1)$ so that $\boldsymbol{G}_-$ varies on the system size. The terms of order ε in (2.25) and (2.26) must, then, be negligible for the $\boldsymbol{G}_-$ parts of $\boldsymbol{B}$ and $\boldsymbol{V}$ dictating $a \approx b$ to have a significant large-scale component in the solution. Consequently the inverse of the second scale, $\lambda_+ \approx (a - a^{-1})/\varepsilon$. Barring the case $a \approx b \approx 1$ (Alfvénic flows – the normalized flow speed of order unity), we observe $\lim_{\varepsilon \to 0} |\lambda_+| = \infty$, i.e., the short scale shrinks to zero. Writing $b/a = 1 + \delta$ [$\delta = O(\varepsilon)$], we can approximate

$$\lambda_- \approx \frac{\delta}{\varepsilon}\left(\frac{1}{a} - a\right)^{-1}, \quad \lambda_+ \approx -\frac{1}{\varepsilon}\left(\frac{1}{a} - a\right). \tag{3.60}$$

Since in the limit $\varepsilon \to 0$, the current $(\nabla \times \boldsymbol{B})$ and the vorticity $(\nabla \times \boldsymbol{V})$ diverge in the small-scale ($\boldsymbol{G}_+$), the MHD cannot ever capture the essence of the two-fluid model. The divergence of these small-scale components implies that the resistive and viscous dissipations can be very large even when resistivity and viscosity coefficient are relatively small.

Let us now derive the condition for which the singular part of the solution vanishes (i.e., $C_+ = 0$). To do this we will relate $C_\pm$ with the energy E and the magnetic helicity H of the fields [23, 24, 27], and express this condition as a relation between E and H. Here we assume that $\boldsymbol{B}$ and $\boldsymbol{V}$ are confined in a simply connected domain (normal components vanish on the boundary). The resulting orthogonality $\int \boldsymbol{G}_- \cdot \boldsymbol{G}_+ \, dx = 0$ and the normalization $\int \boldsymbol{G}_\pm^2 \, dx = 1$ (integral is taken over the total domain) help simplify the analysis. We can evaluate

$$\begin{aligned} E &\equiv \int (\boldsymbol{B}^2 + \boldsymbol{V}^2) \, dx = \alpha_- C_-^2 + \alpha_+ C_+^2, \\ H &\equiv \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx = \frac{C_-^2}{\lambda_-} + \frac{C_+^2}{\lambda_+}, \end{aligned}$$

where $\alpha_\pm = 1 + (\varepsilon\lambda_\pm + a^{-1})^2$. Solving these equations for $C_\pm$, we observe

$$\begin{aligned} C_-^2 &= -\frac{\lambda_-}{D} (E - \alpha_+ \lambda_+ H) \;\to\; \Lambda_- H \quad (\varepsilon \to 0), \\ C_+^2 &= \frac{\lambda_+}{D} (E - \alpha_- \lambda_- H) \\ &\to \frac{1}{(1+a^2)} \left[E - \left(1 + a^{-2}\right) \Lambda_- H \right] \quad (\varepsilon \to 0), \end{aligned}$$

where $D = b(b + a^{-1})(\lambda_+ - \lambda_-)$ and $\Lambda_- = \lim_{\varepsilon \to 0} \lambda_-$ [see (3.60)]. If E and H satisfy the relation

$$E = \left(1 + a^{-2}\right) \Lambda_- H, \tag{3.61}$$

the singular component vanishes ($\lim_{\varepsilon \to 0} C_+ = 0$), and the solution converges to the single Beltrami field, which is the relaxed state of MHD (see Sec. 2.2.1).

$$\boldsymbol{B} = C_- \boldsymbol{G}_-, \quad \boldsymbol{V} = \boldsymbol{B}/a.$$

The energy E satisfying (3.61) is the "minimum energy" accessible for given helicity H and cross helicity $K \equiv \int \boldsymbol{V} \cdot \boldsymbol{B}\, dx$ (we will also discuss in Sec. 4.2). The parameter a is determined by $a + a^{-1} = E/K$.

These relations clearly show that the singular (small scale) part of the double Beltrami field, which can produce a large resistive and viscous dissipations [13], disappears as the field relaxes into the final minimum energy state for given helicity and cross helicity.

3.3.2 Patterns generated by nonlinear dispersive interference

We now explore a mechanism that may create small scales by analyzing the Cauchy characteristics of the incompressible MHD. Eliminating the short time scales of waves (and possible instabilities), we may analyze structures that can persist for longer times. Taking the incompressible limit $C_s^2 \to \infty$ (or $\gamma \to \infty$) in (3.9), the corresponding characteristic equation is obtained as

$$k^2(\boldsymbol{V} \cdot \boldsymbol{k})^2[(\boldsymbol{V} - \boldsymbol{V}_A) \cdot \boldsymbol{k}]^2[(\boldsymbol{V} + \boldsymbol{V}_A) \cdot \boldsymbol{k}]^2 = 0. \tag{3.62}$$

The eikonal $\varphi(\boldsymbol{x})$, where $\boldsymbol{k} = \nabla\varphi(\boldsymbol{x})$, gives the characteristics (rays). The incompressibility closure is sufficient in the present purpose of analysis, and hence, the fast and slow waves degenerate into the shear Alfvén wave and the elliptic part (see Sec. 3.1).

Using the characteristics, we may construct the Cauchy solutions that represent the "propagation" of Cauchy data (initial conditions) along the characteristics (static structures may be regarded as standing waves). Here we encounter two essential problems:

1. Non-integrability (chaos): The characteristic ordinary differential equations (ray equations) may not be integrable.

2. Nonlinearity: The characteristics are determined by the fields themselves (unknown variables).

The first problem does not exist in two-dimensional spaces, where streamlines of divergence-free fields are always integrable. Let us begin with this simple case. The nonlinear two-dimensional analysis can be most simply done following the Grad-Shafranov recipe based on the Clebsch representations of incompressible fields. We have derived the Grad-Shafranov equation of flowing plasmas (3.23) or (3.24) in Sec. 3.2, where $F(\Psi)$ and $P(\Psi)$ are the Cauchy data. Since $F(\Psi)$ and $P(\Psi)$ are arbitrary functions of ψ, they may contain any arbitrary small scale. Parker's model of current sheets can, then, be represented by "wrinkles" in them.

The set of available Cauchy solutions is rather limited because $\Phi = \Phi(\Psi)$ forces the rays to degenerate into a direction that parallels the Ψ contour everywhere. Consequently the two-dimensional projections of $\boldsymbol{V}$ and $\boldsymbol{B}$ are constrained to be parallel severely restricting the class of mathematical available to the physical system. The scope of Cauchy solutions underlying Parker's model of current sheets becomes even narrower when we consider general non-integrable characteristics in three-dimensional systems. When the characteristic curves (magnetic field lines) are embedded densely in a space, inhomogeneous Cauchy data leads to pathology, and the homogeneous Cauchy data yields only the relatively trivial Taylor relaxed state (homogeneous single Beltrami field) with a parallel flow.

Singularities do allow more general solutions with intersecting rays. The essential nature of the "nonlinear interference" produced by ray crossing can be seen in shocks. Since the rays are functions of the fields (variables $\boldsymbol{B}$ and $\boldsymbol{V}$), changes in the ray directions necessarily imply inhomogeneity of the fields. On the contrary, fields must be constant along the rays. One way out of this dilemma is to allow appropriate discontinuities (shocks) at the intersecting points of the rays. Generally, discontinuities may not intersect, and hence, the discontinuous solutions with perpendicular flows (intersecting rays) are still limited.

The restriction of constructing self-consistent solutions is much relaxed in a system with a higher degree of freedom. A singular perturbation, represented by higher-order derivatives, helps to remove the inconsistency produced by ray crossing in models without the singular term. By introducing dispersion, the singular perturbation present in Hall MHD heals discontinuities in MHD. The cooperation of nonlinearity and dispersion (due to the Hall term) determines a specific structure with a characteristic length scale (soliton is an analogy). The small-scale part ($\boldsymbol{G}_+$) of the double Beltrami field is the pattern created by this mechanism. This structure is, thus, singular (infinitely oscillating) in the limit $\varepsilon \to 0$; see (3.60). Vanishing or purely parallel flows (in two dimension, the x-y components of $\boldsymbol{B}$ and $\boldsymbol{V}$ are parallel) will not create this structure, because the singularity is an expression of ray crossing.

It is remarkable that the elementary double Beltrami solution exists in any three dimensional geometry. This is due to the assumption that Beltrami parameters a and b are constant. From (2.25) and (2.26), the divergence-free conditions on $\boldsymbol{B}$ and $\boldsymbol{V}$ demand $(\boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B}) \cdot \nabla a = 0$ and $\boldsymbol{V} \cdot \nabla b = 0$ implying that the Beltrami parameters are the required Cauchy data – they are assumed to be constant, and hence, the double Beltrami fields are robust in chaotic characteristics. This is in marked contrast to Parker's model that considers wrinkles of Cauchy data as the origin of small scales. In the double Beltrami field, the inhomogeneity stems from nonlinear dispersive interference produced by the intersections of the ideal characteristic curves.

Chapter 4

Stability of flowing plasmas

4.1 Lyapunov stability analysis

Stability of a plasma with a shear flow constitutes a very challenging problem because the interaction between the perturbations and the ambient flow cannot generally be cast in an appropriate Hamiltonian form. The linear operator of flow dynamics become "non-Hermitian" due to the fact the energy (conjugate to "time") corresponds to the frequency of perturbations, and the frequency of perturbations in a flow may assume complex values.

The most essential tool in the stability analysis of a "no-flow" (static) plasma is the "energy principle" deduced from the Hermicity (self-adjointness) of the generating operator of the linearized ideal MHD equations [4]. For the Hermitian operator, the von Neumann theorem assures spectral decomposition. In other words, the time derivative (∂_t) can be replaced by an eigenmode ($-i\omega$), allowing a stability condition to be obtained by solving an eigenvalue problem (dispersion relation). For the non-Hermitian operator, however, spectral decomposition is not assured. Without this merit of mathematical structure, the stability analysis of a flowing plasma encounters a fatal problem. Even if all eigenvalues ω are real, stability is not guaranteed. That is, non-exponential (algebraic) instabilities can exist [7, 8, 9, 10, 11]. The standard spectral analysis cannot provide a complete understanding of stability in a non-Hermitian system, in which the standard notion of energy (Hamiltonian) does not pertain and the energy may cease to be the basic determinant of the stability of the flow (it is even difficult to define an appropriate energy for perturbations).

The stability analysis of flowing plasmas, then, requires a wider framework. The notion of a Lyapunov function is a natural extension of Hamiltonian. One important idea was introduced by Arnold for a two-dimensional flow [54, 55, 56]. In this analysis we have to give up on tracing the exact orbits of dynamics. Instead of a detailed analysis, however, we may find a bound for the orbits, and this may provide us with a sufficient condition of stability.

In this chapter, we study the stability of a special class of states with flows that are derivable from variational principles. These states are considered as relaxed states, however the notion of relaxed states does not automatically warrant stability. For this special class of flows, we will show the existence of a constant of motion (Lyapunov function) that bounds the energy of perturbations under some appropriate conditions. The "coerciveness" in the topology of the energy norm is required to get a sufficient condition for stability.

4.1.1 Variational principle and constant of motion

We first cast the method in an abstract theorem, which gives a relation between a variational principle characterizing an equilibrium and a constant of motion for perturbations [57].

Let $f(a, b)$ be a bilinear map. We define $\mathcal{F}(u) = f(u, u)$, and consider an abstract nonlinear evolution equation

$$\partial_t u = \mathcal{F}(u). \tag{4.1}$$

We further suppose that there are symmetric bilinear forms $h_j(a, b)$ $(j =$

$1, \ldots, \nu)$ such that

$$h_j(u, \mathcal{F}(u)) = 0 \quad (j = 1, \ldots, \nu, \ \forall u). \tag{4.2}$$

It is now easy to show that $H_j(u) = h_j(u, u)$ [u is a solution of (4.1)] is a constant of motion for the evolution equation (4.1);

$$\begin{aligned} \frac{d}{dt} H_j(u) &= 2h_j(u, \partial_t u) \\ &= 2h_j(u, \mathcal{F}(u)) = 0. \end{aligned} \tag{4.3}$$

Let u_0 be a stationary point (equilibrium) of (4.1), i.e., $\mathcal{F}(u_0) = 0$. We assume that u_0 solves

$$\delta \left[\sum_{j=1}^{\nu} \mu_j H_j(u) \right] = 0 \tag{4.4}$$

with some fixed real numbers μ_j $(j = 1, \ldots, \nu)$. We call such a u_0 as a "Beltrami field."

Remark 1. If (4.4) has a unique (or isolated) solution u_0, then this u_0 is an equilibrium of (4.1). Indeed, any departure from u_0 will change the value of $G(u) \equiv \sum_{j=1}^{\nu} \mu_j H_j(u)$, while $G(u)$ is a constant of motion.

To study the perturbations around u_0, the following theorem plays an essential role.

Theorem 1. *Suppose that* $u = u_0 + \widetilde{u}$ *(*u_0 *is a Beltrami field) satisfies either (4.1) or its "linearized" equation*

$$\partial_t \widetilde{u} = f(u_0, \widetilde{u}) + f(\widetilde{u}, u_0). \tag{4.5}$$

Then,

$$G(\widetilde{u}) = \sum_{j=1}^{\nu} \mu_j H_j(\widetilde{u}) \tag{4.6}$$

is a constant of motion.

(proof) Using (4.2), we observe

$$\begin{aligned} 0 &= \sum \mu_j h_j(u, \mathcal{F}(u)) \\ &= \sum \mu_j h_j(u_0 + \widetilde{u}, \mathcal{F}(u_0 + \widetilde{u})) \\ &= \sum \mu_j h_j(u_0, \mathcal{F}(u_0 + \widetilde{u})) \\ &\qquad + \sum \mu_j h_j(\widetilde{u}, \mathcal{F}(u_0 + \widetilde{u})). \end{aligned} \tag{4.7}$$

Since (4.4) implies $\sum \mu_j h_j(u_0, \delta) = 0$ $(\forall \delta)$, the first sum in (4.7) vanishes. Hence, if u solves (4.1), we obtain

$$\begin{aligned} \frac{d}{dt} G(\widetilde{u}) &= 2 \sum \mu_j h_j(\widetilde{u}, \partial_t \widetilde{u}) \\ &= 2 \sum \mu_j h_j(\widetilde{u}, \mathcal{F}(u_0 + \widetilde{u})) = 0. \end{aligned} \tag{4.8}$$

We can rewrite (4.7) as

$$\begin{aligned} 0 &= \sum \mu_j h_j(\widetilde{u}, f(u_0, \widetilde{u}) + f(\widetilde{u}, u_0)) \\ &\qquad + \sum \mu_j h_j(\widetilde{u}, \mathcal{F}(\widetilde{u})). \end{aligned} \tag{4.9}$$

By (4.2), the second term of (4.9) vanishes. If $\widetilde{u}$ is a solution of (4.5), we obtain

$$\frac{d}{dt} G(\widetilde{u}) = 2 \sum \mu_j h_j(\widetilde{u}, f(u_0, \widetilde{u}) + f(\widetilde{u}, u_0)) = 0. \tag{4.10}$$

□

We note that although each functional H_j occurring in the sum that defines G is a constant of motion for the total field u, it is only the special linear combination (4.6) that is conserved for the perturbation, $\widetilde{u}$. The coefficients μ_j included in G are the structure (Beltrami) parameters characterizing the equilibrium.

If a continuous quadratic form $F(v)$ satisfies (on a Hilbert space V)

$$F(v) \geq c\|v\|^2 \quad (\forall v \in V) \tag{4.11}$$

with some positive constant c ($\|v\|$ is the norm of v in V), $F(v)$ is said to be "coercive."

Obviously, we have the following Proposition.

Proposition 1. *If* $G(v) = \sum_{j=1}^{\nu} \mu_j H_j(v)$ *with given* μ_j *is a coercive form, then*

1. $G(u)$ *has a unique "minimizer" that is given by the variational principle (4.4),*
2. *the minimizer* u_0 *of* $G(u)$ *is a stationary point (equilibrium) of (4.1),*
3. *the minimizer* u_0 *is "stable"; the norm of every perturbation* $\widetilde{u}$ *is bounded by a constant that depends upon* $G(\widetilde{u}|_{t=0})$.

4.1.2 Two-dimensional vortex dynamics

We give a simple particular example of vortex dynamics in two-dimensional space, (where we use the term "two-dimensional" as x-y plane in the Cartesian coordinate). A two-dimensional (shallow) incompressible flow in a bounded domain Ω obeys the vortex dynamics

$$\partial_t W + \{\Phi, W\} = 0, \tag{4.12}$$

where the stream function Φ defined by $\boldsymbol{v} = (\partial_y \Phi, -\partial_x \Phi)$ acts as the effective Hamiltonian, W $(= -\Delta\Phi)$ is the flow vorticity, and the Poisson bracket has the standard form

$$\begin{aligned}\{a, b\} &= (\partial_y a)(\partial_x b) - (\partial_x a)(\partial_y b) \\ &= -(\nabla a \times \nabla b) \cdot \boldsymbol{e}_z, \end{aligned} \tag{4.13}$$

with $\boldsymbol{e}_z = \nabla x \times \nabla y$. The circulation of the flow must be conserved (Kelvin's theorem);

$$\oint_\Gamma \boldsymbol{n} \cdot \nabla\Phi \; d\gamma = K \text{ (given constant)}, \tag{4.14}$$

where $\boldsymbol{n}$ is the unit normal vector onto the boundary Γ. To confine the flow $\boldsymbol{v} = \nabla\Phi \times \boldsymbol{e}_z$ in Ω we demand

$$\Phi|_\Gamma = C \text{ (unknown constant)}, \tag{4.15}$$

where $|_\Gamma$ denotes the trace to the boundary value.

The general stationary solution (equilibrium flow) of this dynamics is given by $\{\Phi, W\} = 0$ implying $W = w(\Phi)$ with w being a certain smooth function. For the simplest nontrivial choice (w linear in Φ), the equilibrium condition yields what is called the "Beltrami flow,"

$$-\Delta\Phi(= W) = \mu\Phi \quad (\mu = \text{real constant}). \tag{4.16}$$

Remark 2. The Beltrami equation (4.16) with the circulation and boundary conditions (4.14) - (4.15) is equivalent to an inhomogeneous equation: writing

$\Phi = \phi + C$ (C is a constant), the transformed problem reads

$$(-\Delta - \mu)\phi = \mu C,$$
$$\phi|_\Gamma = 0, \quad \oint_\Gamma \boldsymbol{n} \cdot \nabla\phi \, d\gamma = K.$$

If μ is the eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition, a solution may be obtained by demanding $C = 0$. Otherwise, the system leads to $\phi = -\mu(\Delta + \mu)^{-1}C$ with the constant C chosen to yield the prescribed K. We, thus, have a nontrivial solution for every complex number μ; the point spectrum of the Laplacian operator with the inhomogeneous circulation and boundary conditions (4.14) - (4.15) spans the totality of complex numbers (see *Remark* in Sec. 2.2). In what follows, we assume that μ is a real number (then ϕ is a real function).

The evolution equation (4.12), under the circulation and boundary conditions (4.14) - (4.15), has two essential integrals (constants of motion):

$$H_0 = \|W\|^2 \equiv \int_\Omega |W|^2 \, dx \quad \text{(enstrophy)}, \tag{4.17}$$

$$\begin{aligned} H_1 &= \|\nabla\Phi\|^2 \\ &= (W, \mathcal{P}\Phi) \equiv \int_\Omega W \cdot \mathcal{P}\Phi \, dx \quad \text{(energy)}, \end{aligned} \tag{4.18}$$

where $\mathcal{P}\Phi = \Phi - C$ (C is chosen so that $\mathcal{P}\Phi|_\Gamma = 0$) is a projection to homogenize the boundary condition (4.15). It is straightforward to see that the Beltrami equation (4.16) is reproduced as the Euler-Lagrange equation of the variational principle

$$\delta(H_0 - \mu H_1) = 0 \tag{4.19}$$

with the circulation and boundary conditions (4.14) - (4.15).

To study the stability of a Beltrami flow (denote the Hamiltonian by Φ_0), we linearize (4.12) with writing $\Phi = \Phi_0 + \varphi$ and $-\Delta\varphi = \omega$ (the circulation $\int_\Omega \omega dx$ must be zero);

$$\partial_t \omega + \{\Phi_0, \omega\} + \{\varphi, -\Delta\Phi_0\} = 0. \tag{4.20}$$

Using the equilibrium (4.16), we can write

$$\partial_t \omega + \{\Phi_0, \omega - \mu\varphi\} = 0. \tag{4.21}$$

We easily verify that

$$G(\varphi) = (\omega, \omega - \mu\mathcal{P}\varphi) = \|\omega\|^2 - \mu\|\nabla\varphi\|^2 \tag{4.22}$$

is a constant of motion ($dG(\varphi)/dt = 0$) associated with the linearized dynamics (4.21). Due to **Theorem 1**, the variation (4.19) implies that the integral $G(\varphi)$ is a constant of motion for φ satisfying the nonlinear equation (4.12), too. In a bounded domain, we have the (Poincaré type) inequality

$$\| - \Delta\varphi\|^2 \geq \lambda\|\nabla\varphi\|^2 \tag{4.23}$$

with λ being the smallest eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition (one easily finds $\lambda > 0$). We, thus, have

$$G(\varphi) \geq \left(1 - \frac{\mu}{\lambda}\right) \| - \Delta\varphi\|^2, \tag{4.24}$$

or

$$(\lambda - \mu)\|\nabla\varphi\|^2 \leq G(\varphi), \tag{4.25}$$

implying that the energy $\|\nabla\varphi\|^2$ remains bounded for $\mu < \lambda$, because $G(\varphi)$ is a constant determined by the initial condition of the perturbation φ; the bound $\mu < \lambda$ on the Beltrami parameter gives a sufficient condition for the stability of the Beltrami flow. The functional G of (4.24) for $\mu < \lambda$ is a coercive form (we consider G as a continuous form in the topology of H^2 Sobolev space), see (4.11).

4.2 Stability of single Beltrami flows

4.2.1 Variational principle and stability condition of MHD

We shall now put to work the general mathematical framework developed in previous section. We will apply **Theorem 1** and **Proposition 1** to investigate the stability of a three dimensional plasma equilibrium with a flow.

Ideal (incompressible) MHD description of a plasma is governed by the equations (2.7) - (2.8) with $\varepsilon = 0$; the force equation

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{4.26}$$

and the induction equation

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \tag{4.27}$$

We assume boundary conditions

$$\boldsymbol{n} \cdot \boldsymbol{V} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{B} = 0 \quad \text{on } \Gamma, \tag{4.28}$$

and flux conditions

$$\int_{\Sigma_\ell} \boldsymbol{n} \cdot \boldsymbol{B} \, ds = K_\ell \quad (\ell = 1, \dots, m), \tag{4.29}$$

where the fluxes through the cuts are given constants. The dynamics allows three important constants of motion:

$$H_0 = \|\boldsymbol{B}\|^2 + \|\boldsymbol{V}\|^2 \equiv \int_\Omega \boldsymbol{B}^2 + \boldsymbol{V}^2 \, dx \quad \text{(energy)}, \tag{4.30}$$

$$H_1 = (\boldsymbol{A}, \boldsymbol{B}) \equiv \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} \, dx \quad \text{(magnetic helicity)}, \tag{4.31}$$

$$H_2 = 2(\boldsymbol{V}, \boldsymbol{B}) \quad \text{(cross helicity)}, \tag{4.32}$$

where $\boldsymbol{A}$ is the vector potential.

The variational principle

$$\delta(H_0 - \mu_1 H_1 - \mu_2 H_2) = 0 \tag{4.33}$$

gives Beltrami fields defined by

$$(1 - \mu_2^2)\nabla \times \boldsymbol{B} = \mu_1 \boldsymbol{B}, \tag{4.34}$$

$$\boldsymbol{V} = \mu_2 \boldsymbol{B}, \tag{4.35}$$

see (2.22) - (2.23).

Due to **Theorem 1**, the integral

$$G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}}) = \|\widetilde{\boldsymbol{B}}\|^2 + \|\widetilde{\boldsymbol{V}}\|^2 - \mu_1(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}) - 2\mu_2(\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{B}}) \tag{4.36}$$

is a constant of motion for the perturbations $\widetilde{\boldsymbol{B}}$ $(= \nabla \times \widetilde{\boldsymbol{A}})$ and $\widetilde{\boldsymbol{V}}$ satisfying the nonlinear equation (4.26) - (4.27), or their linearized equations. The flux condition (4.29) demands $\widetilde{\boldsymbol{B}} \in L^2_\Sigma(\Omega)$.

We now prove the inequality

$$(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}) \leq |\lambda|^{-1} \|\widetilde{\boldsymbol{B}}\|^2, \tag{4.37}$$

where $|\lambda| = \min_j |\lambda_j|$ [λ_j $(j = 1, 2, \ldots)$ are the eigenvalues of the self-adjoint curl operator]. Invoking the spectral resolution theorem due to Yoshida-Giga [37], we expand $\boldsymbol{u} = \sum(\boldsymbol{u}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j$ $(\forall \boldsymbol{u} \in L^2_\Sigma(\Omega))$, where $\boldsymbol{\psi}_j$ is the eigenfunction of the self-adjoint curl operator belonging to an eigenvalue λ_j, and write

$$\widetilde{\boldsymbol{B}} = \sum (\widetilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j,$$

and

$$\mathcal{P}\widetilde{\boldsymbol{A}} = \sum (\widetilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j/\lambda_j,$$

where $\mathcal{P}$ is the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$. We observe

$$\begin{aligned}
(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}) &= (\mathcal{P}\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}) \\
&\leq \|\mathcal{P}\widetilde{\boldsymbol{A}}\| \cdot \|\widetilde{\boldsymbol{B}}\| \\
&= \left[\sum (\widetilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2/\lambda_j^2\right]^{-1/2} \left[\sum (\widetilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2\right]^{-1/2} \\
&\leq |\lambda|^{-1} \sum (\widetilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2 \\
&= |\lambda|^{-1} \|\widetilde{\boldsymbol{B}}\|^2.
\end{aligned}$$

Using

$$2(\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{B}}) \leq \alpha\|\widetilde{\boldsymbol{V}}\|^2 + \alpha^{-1}\|\widetilde{\boldsymbol{B}}\|^2 \quad (\forall \alpha > 0),$$

we obtain

$$G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}}) \geq \left(1 - \frac{|\mu_2|}{\alpha} - \frac{|\mu_1|}{|\lambda|}\right) \|\widetilde{\boldsymbol{B}}\|^2 + (1 - \alpha|\mu_2|)\, \|\widetilde{\boldsymbol{V}}\|^2. \tag{4.38}$$

The choices $\alpha = 1/|\mu_2|$, and $\alpha = |\mu_2|/(1 - |\mu_1|/|\lambda|)$ convert (4.38) to

$$G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}}) \geq \left(1 - \mu_2^2 - \frac{|\mu_1|}{|\lambda|}\right) \|\widetilde{\boldsymbol{B}}\|^2, \tag{4.39}$$

and

$$G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}}) \geq \left(1 - \frac{\mu_2^2}{1 - |\mu_1|/|\lambda|}\right) \|\widetilde{\boldsymbol{V}}\|^2, \tag{4.40}$$

respectively. If $1 - \mu_2^2 - |\mu_1|/|\lambda| > 0$, then (4.39) and (4.40) give bounds for the energy associated with the magnetic ($\widetilde{\boldsymbol{B}}$) as well as the velocity ($\widetilde{\boldsymbol{V}}$) fluctuations.

The "sufficient condition" for the stability, therefore, consists of the simultaneous inequalities

$$\mu_2^2 \;<\; 1, \tag{4.41}$$

$$\sigma \equiv \frac{|\mu_1|}{1 - \mu_2^2} \;<\; |\lambda|, \tag{4.42}$$

where σ stands for the eigenvalue of the Beltrami equation (4.34) for $\mu_1 > 0$. The first stability condition requires that the flow velocity must not exceed the local Alfvén speed [see (4.35)], while the second condition demands that σ must not exceed the minimum of $|\lambda_j|$ (λ_j is the eigenvalue of the self-adjoint curl operator).

Combining a constant of motion and a coerciveness relation, we have derived a bound for the energy of perturbations yielding a sufficient condition for stability. The constant of motion (**Theorem 1**) is closely related to the variational principle characterizing the Beltrami equilibrium. Under appropriate boundary conditions, coerciveness is measured by the highest order of derivatives included in the functional, which is a consequence of a Poincaré-type inequality. In the inequality (4.37), the constants are related to the eigenvalues of the self-adjoint curl operator assuming sufficiently smooth functions. These eigenvalues are compared with the Beltrami parameter to determine the stability in (4.42). The constant of motion may be regarded as a "Lyapunov function." Tasso [58, 59] developed a similar scheme for a dissipative system where the corresponding Lyapunov function may decay implying the damping of perturbations; see also [60]. The key, again, is the "coerciveness" relation. We can apply (4.37) and similar inequalities to quantify the bound in terms of the eigenvalue of the self-adjoint curl operator.

If we consider only exponential instabilities (replacing ∂_t by $-i\omega$ and studying the dispersion relations), we can develop a more detailed analysis for special geometries. For example, in a 1-D slab, the necessary and sufficient condition for stability against exponential growth is that either (4.41) or (4.42) is satisfied (not "and") [61]. In this system the magnetic field curvature that may destabilize the Alfvén waves (kink modes) is absent, and only possible instabilities are of the Kelvin-Helmholtz type. Without the magnetic field, the well-known stability criterion for the Kelvin-Helmholtz mode is precisely our condition (4.42). The magnetic field, in this case, has a stabilizing effect (because of the absence of the kink modes), and hence, if (4.41) is satisfied, even when (4.42) is violated, the system is stable.

4.2.2 Coercive form and convex form

The method developed here differs from the standard argument for stability based on the second variation of the target functional (constant of motion) [62] based on Arnold's method [54, 55, 56]. If an equilibrium is defined by a variational principle (first variation= 0), the stability of the stationary point may be examined by analyzing the spectrum of the "Hessian" of the target functional on a function space. In general, this problem is highly nontrivial because the linearized operator describing the dynamics of perturbations may be non-Hermitian. When the target functional of the variational principle is a symmetric quadratic form, however, the second variation yields a symmetric Hessian. This is an essential characteristic of the "Beltrami" class of equilibria, and it greatly simplifies the stability analysis. Our method does not invoke the second variation. Instead, we have found a constant of motion that is naturally deduced from the variational principle characterizing the stationary point. The success of this method is also primarily due to the assumption that the target functional (G = linear combination of constants of motion) is a symmetric quadratic form. The key of the stability theory is, then, the coerciveness of the constant of motion; it allows us to put a bound on the perturbation norm. The constant of motion of a perturbation is formally equivalent to the target functional G of the variational principle that determines the stationary point (**Theorem 1**). The coerciveness demands that the G is a convex form, and hence, the coerciveness condition may be related to the index of the Hessian. The former is, however, a more fundamental notion that is directly related to the "topology" of the function space. Next, we examine the relation between the coerciveness and the index of the Hessian.

We consider the target functional [see (4.30) - (4.33)]

$$
\begin{aligned}
G(\boldsymbol{B}, \boldsymbol{V}) &= H_0 - \mu_1 H_1 - \mu_2 H_2 \\
&= \|\boldsymbol{B}\|^2 + \|\boldsymbol{V}\|^2 - \mu_1(\boldsymbol{A}, \boldsymbol{B}) - 2\mu_2(\boldsymbol{V}, \boldsymbol{B}).
\end{aligned}
$$

The simplest representation of the functional and its Hessian operator (second variation of G) may be obtained in terms of the eigenfunction of the self-adjoint curl operator [37]. The complete orthogonal set $\boldsymbol{\psi}_j$ ($\nabla \times \boldsymbol{\psi}_j = \lambda_j \boldsymbol{\psi}_j$) spanning

$L^2_\Sigma(\Omega)$ allows

$$\begin{aligned}\boldsymbol{B} &= \sum_j b_j \boldsymbol{\psi}_j + \boldsymbol{B}_h,\\ \boldsymbol{V} &= \sum_j v_j \boldsymbol{\psi}_j + \boldsymbol{V}_h,\end{aligned}$$

where $\boldsymbol{B}_h$ and $\boldsymbol{V}_h$ are harmonic fields (see *Remark* in Sec. 2.2). The flux condition (4.29) determines $\boldsymbol{B}_h$, while $\boldsymbol{V}_h$ is an unknown variable. Defining $\boldsymbol{B} - \boldsymbol{B}_h = \nabla \times \boldsymbol{A}_\sigma$ ($\boldsymbol{n} \times \boldsymbol{A}_\sigma = 0$ on Γ) and $\boldsymbol{A}_g = \boldsymbol{A} - \boldsymbol{A}_\sigma$, we may write $\boldsymbol{B}_h = \nabla \times \boldsymbol{A}_g$. Using the definition

$$\Lambda_j = (\boldsymbol{\psi}_j, \boldsymbol{A}_g) \quad \text{(given constants)},$$

along with the other expansions, we obtain

$$\begin{aligned}H_0 &= \|\boldsymbol{B}\|^2 + \|\boldsymbol{V}\|^2 = \sum_j \left(b_j^2 + v_j^2\right) + \|\boldsymbol{B}_h\|^2 + \|\boldsymbol{V}_h\|^2,\\ H_1 &= (\boldsymbol{A}, \boldsymbol{B}) = \sum_j \left(\lambda_j^{-1} b_j^2 + 2 b_j \Lambda_j\right) + (\boldsymbol{A}_g, \boldsymbol{B}_h),\\ H_2 &= 2(\boldsymbol{V}, \boldsymbol{B}) = 2\sum_j v_j b_j + 2\left(\boldsymbol{V}_h, \boldsymbol{B}_h\right),\end{aligned}$$

and

$$\begin{aligned}G(\boldsymbol{B}, \boldsymbol{V}) &= \sum_j \left(b_j^2 + v_j^2 - \mu_1 \lambda_j^{-1} b_j^2 - 2\mu_1 b_j \Lambda_j - 2\mu_2 b_j v_j\right)\\ &\quad + \|\boldsymbol{B}_h\|^2 + \|\boldsymbol{V}_h\|^2 - 2\mu_2\left(\boldsymbol{B}_h, \boldsymbol{V}_h\right) - \mu_1(\boldsymbol{A}_g, \boldsymbol{B}_h).\end{aligned}$$

We can now calculate the variations explicitly. The first variation (under $\delta \boldsymbol{A}_g = 0$; gauge invariance)

$$\begin{aligned}\delta G &= \sum_j \left[2\left(b_j - \mu_1 \lambda_j^{-1} b_j - \mu_2 v_j - \mu_1 \Lambda_j\right)\delta b_j\right. \\ &\qquad \left. +2\left(v_j - \mu_2 b_j\right)\delta v_j\right] + 2\left(\boldsymbol{V}_h - \mu_2 \boldsymbol{B}_h\right)\cdot \delta \boldsymbol{V}_h\end{aligned}$$

determines the stationary point (equilibrium);

$$\begin{aligned}\boldsymbol{V}_{0h} &= \mu_2 \boldsymbol{B}_h, \quad v_{0j} = \mu_2 b_{0j}, \\ b_{0j} &= \frac{\mu_1 \Lambda_j}{\left(1 - \mu_2^2 - \mu_1 \lambda_j^{-1}\right)},\end{aligned}$$

where $\boldsymbol{B}_h$ and Λ_j are given constants.

Denoting

$$v_j = v_{0j} + \widetilde{v}_j, \quad b_j = b_{0j} + \widetilde{b}_j,$$

and defining the linear-transformed variables

$$\widetilde{c}_j = \widetilde{v}_j - \mu_2 \widetilde{b}_j,$$

we obtain

$$\begin{aligned}G &= \sum_j \left(\widetilde{b}_j^2 + \widetilde{v}_j^2 - \mu_1 \lambda_j^{-1} \widetilde{b}_j^2 - 2\mu_2 \widetilde{b}_j \widetilde{v}_j\right) + \|\widetilde{\boldsymbol{V}}_h\|^2 + G_0 \\ &= \sum_j \left[\widetilde{c}_j^2 + \left(1 - \mu_2^2 - \mu_1 \lambda_j^{-1}\right)\widetilde{b}_j^2\right] + \|\widetilde{\boldsymbol{V}}_h\|^2 + G_0,\end{aligned}$$

where

$$G_0 = \left(1 - \mu_2^2\right) \|\boldsymbol{B}_h\|^2 - \mu_1(\boldsymbol{A}_g, \boldsymbol{B}_h)$$

is the minimum value of G. This expression of G gives the "diagonalized" Hessian evaluated at the stationary point. Let us write $\widetilde{\boldsymbol{V}}_h = \widetilde{v}_h \boldsymbol{h}$ ($\boldsymbol{h}$ is the normalized harmonic field) and define $\widetilde{c}_h = \widetilde{v}_h$ (the harmonic magnetic field $\boldsymbol{B}_h$ is fixed by the flux condition). The independent degrees of freedom associated with perturbations may be represented by an infinite dimension vector

$$\widetilde{\boldsymbol{u}} = (\widetilde{c}_h, \widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{b}_1, \widetilde{b}_2, \ldots),$$

which lets us cast G in the canonical form

$$G = \widetilde{\boldsymbol{u}} \mathcal{D} \widetilde{\boldsymbol{u}}^T + G_0$$

with the diagonalized Hessian

$$\mathcal{D}_{j,k} = \begin{cases} \delta_{j,k} & \text{acting on } \widetilde{c}_h, \widetilde{c}_1, \widetilde{c}_2, \ldots \ , \\ (1 - \mu_2^2 - \mu_1 \lambda_j^{-1}) \delta_{j,k} & \text{acting on } \widetilde{b}_1, \widetilde{b}_2, \ldots \ , \end{cases} \tag{4.43}$$

where $\delta_{j,k} = 1$ (or 0) when $j = k$ (or $j \neq k$). The stationary point is stable if the index of the Hessian (the number of the negative eigenvalues of $\mathcal{D}$) is zero. The sufficient conditions for stability can be directly read off from (4.43), and are

$$\mu_2^2 < 1 \tag{4.44}$$

and

$$
\begin{cases}
\dfrac{\mu_1}{1-\mu_2^2} < \lambda_+ & \text{if } \mu_1 > 0, \\
\dfrac{|\mu_1|}{1-\mu_2^2} < |\lambda_-| & \text{if } \mu_1 < 0,
\end{cases}
\tag{4.45}
$$

with $\lambda_+ = \min_{\lambda_j>0} \lambda_j$ and $\lambda_- = \max_{\lambda_j<0} \lambda_j$ [hence, $|\lambda| = \min(\lambda_+, |\lambda_-|)$]. Comparing (4.44) and (4.45) with (4.41) and (4.42), we find a slight improvement of the stability bound. This is due to the exact evaluation of the Hessian.

4.3 Stability of double Beltrami flows

4.3.1 Non-coercive form in two-fluid MHD

In this section, we consider the stability of an ideal two-fluid (Hall) MHD plasma [63]. A relaxed state derived from a variational principle is expressed by the double Beltrami field discussed in Sec. 2.2.2. However, the two-fluid model, including a singular perturbation (primarily given by the Hall term), leads to a relaxed state that may not be recognized as the minimization of energy unlike Taylor relaxed (single Beltrami) state [18, 19], since the free energy is not bounded (that is, not convex) in the energy norm [27]. It means that, unlike one-fluid MHD in previous section, the functional G (linear combination of constants of motion) of two-fluid MHD is not a coercive nor a convex form due to the singular perturbation term included in the ion helicity. This affects the stability analysis by Lyapunov function. Since the functional is not a coercive (nor convex) form, Arnold's method is no longer effective in this system, as suggested by Holm [26]. The singular perturbation demands a "stronger topology" of the functions space to give bounds for all possible fluctuations. Therefore, in order to bound the fluctuation energy, we have to find a stronger (higher-order derivative) constant. It will be shown soon that an enstrophy order constant is required for the two-fluid MHD stability. Although the enstrophy is not a constant of motion in general, we can make a precise theory for two special cases under assumption of a symmetry and derive a sufficient condition of linear stability.

We consider an ideal two-fluid plasma in a domain Ω. The governing equations are (2.7) - (2.8), where we set $\varepsilon = 1$ for simplicity;

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{4.46}$$

$$\partial_t \boldsymbol{B} = \nabla \times [(\boldsymbol{V} - \nabla \times \boldsymbol{B}) \times \boldsymbol{B}] = 0. \tag{4.47}$$

We assume boundary conditions

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{n} \cdot (\nabla \times \boldsymbol{u}) = 0 \quad \text{on } \Gamma, \tag{4.48}$$

where $\boldsymbol{u} = \boldsymbol{B}$ or $\boldsymbol{V}$ and $\boldsymbol{n}$ is a unit normal vector onto Γ.

The dynamics allows three important constants of motion;

$$H_0 = \|\boldsymbol{B}\|^2 + \|\boldsymbol{V}\|^2 \qquad \text{(energy)}, \tag{4.49}$$

$$H_1 = (\boldsymbol{A}, \boldsymbol{B}) \qquad \text{(magnetic helicity)}, \tag{4.50}$$

$$H_2 = (\boldsymbol{A} + \boldsymbol{V}, \boldsymbol{B} + \nabla \times \boldsymbol{V}) \qquad \text{(helicity of ion fluid)}. \tag{4.51}$$

Note that, without the second condition of (4.48), the ion flow helicity H_2 does not conserve.

The variational principle

$$\delta(H_0 + a^{-1}H_1 - b^{-1}H_2) = 0 \tag{4.52}$$

leads the double Beltrami field

$$\boldsymbol{B} = a(\boldsymbol{V} - \nabla \times \boldsymbol{B}), \tag{4.53}$$

$$\boldsymbol{B} + \nabla \times \boldsymbol{V} = b\boldsymbol{V}. \tag{4.54}$$

It is a positive fact that the double Beltrami field equations (4.53) and (4.54) are the Euler-Lagrange equations of the variation (4.52), but it does not necessarily require the field to be a minimizer of the energy H_0 for given helicities H_1 and H_2 [27].

Due to **theorem 1**, the variation (4.52) implies that the integral

$$\begin{aligned} G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}}) &= \|\widetilde{\boldsymbol{B}}\|^2 + \|\widetilde{\boldsymbol{V}}\|^2 + a^{-1}(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}) \\ & \quad -b^{-1}(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{B}} + \nabla \times \widetilde{\boldsymbol{V}}) \end{aligned} \tag{4.55}$$

is a constant of motion for the perturbations $\widetilde{\boldsymbol{B}}$ ($= \nabla \times \widetilde{\boldsymbol{A}}$) and $\widetilde{\boldsymbol{V}}$ satisfying the nonlinear equation (4.46) - (4.47), or its linearized equation. However, reflecting the fact that the functional $G = H_0 + a^{-1}H_1 - b^{-1}H_2$ is not a coercive form in the energy (L^2) norm because of the term $(\boldsymbol{V}, \nabla \times \boldsymbol{V})$ in H_2 [26, 27], this constant of motion $G(\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{V}})$ does not yield a bound for energy of perturbations. It is this point that makes the present case difficult and different from previous cases. In order to discuss the stability, we have to first find a higher-order (in derivatives) than $(\boldsymbol{V}, \nabla \times \boldsymbol{V})$ and positive definite (convex) constant of motion. Therefore, at least, an enstrophy ($\|\nabla \times \boldsymbol{V}\|^2$) order invariant is required. For two special classes of the double Beltrami fields, we can find appropriate conservation laws that each yield a Lyapunov stability condition.

4.3.2 Two-dimensional dynamics

We consider a cylindrical geometry of arbitrary cross-section and assume that the equilibrium magnetic field $\boldsymbol{B}_0$ and velocity field $\boldsymbol{V}_0$ are homogeneous along the axis of the cylinder, which we set z in the Cartesian coordinate x-y-z (that is, $\partial_z = 0$). Using a flux function ψ_0 and a stream function ϕ_0, we may write

$$\boldsymbol{B}_0 = \nabla\psi_0(x,y) \times \nabla z + B_{z0}(x,y)\nabla z, \tag{4.56}$$

$$\boldsymbol{V}_0 = \nabla\phi_0(x,y) \times \nabla z + V_{z0}(x,y)\nabla z. \tag{4.57}$$

The boundary condition (4.48) demands that ψ_0, ϕ_0, B_{z0} and V_{z0} are constants on the boundary Γ and we assume a periodic boundary with respect to z. Substituting (4.56) - (4.57) in (4.53) - (4.54), we get

$$\begin{cases} B_{z0} = \phi_0 - a^{-1}\psi_0 + C_B, \\ V_{z0} = b\phi_0 - \psi_0 + C_V, \\ \Delta\psi_0 = (a^{-1} - b)\phi_0 + (1 - a^{-2})\psi_0 + a^{-1}C_B - C_V, \\ \Delta\phi_0 = (1 - b^2)\phi_0 + (b - a^{-1})\psi_0 + C_B - bC_V, \end{cases} \tag{4.58}$$

where C_B and C_V are constants and correspond to external fields in the z-direction. Assuming strong external z-field, we consider perturbations depend on x, y and t (we neglect z-dependence) [64, 65, 66]. In addition, for an incompressible motion, the perturbed fields $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{V}}$ allow a Clebsch-like representation,

$$\widetilde{\boldsymbol{B}} = \nabla\widetilde{\psi}(x,y,t) \times \nabla z + \widetilde{B}_z(x,y,t)\nabla z, \tag{4.59}$$

$$\widetilde{\boldsymbol{V}} = \nabla\widetilde{\phi}(x,y,t) \times \nabla z + \widetilde{V}_z(x,y,t)\nabla z. \tag{4.60}$$

Writing $\psi = \psi_0 + \widetilde{\psi}$, $\phi = \phi_0 + \widetilde{\phi}$, $B_z = B_{z0} + \widetilde{B}_z$ and $V_z = V_{z0} + \widetilde{V}_z$, the two-fluid (Hall) MHD equations (4.46) and (4.47) can be cast in a form of coupled nonlinear Liouville equations;

$$\partial_t(-\Delta\phi) + \{\phi, -\Delta\phi\} - \{\psi, -\Delta\psi\} = 0, \tag{4.61}$$

$$\partial_t V_z + \{\phi, V_z\} - \{\psi, B_z\} = 0, \tag{4.62}$$

$$\partial_t \psi + \{\phi - B_z, \psi\} = 0, \tag{4.63}$$

$$\partial_t B_z + \{\phi, B_z\} - \{V_z + \Delta\psi, \psi\} = 0, \tag{4.64}$$

where we use the Poisson bracket defined in (4.13). In this system, corresponding to (4.55),

$$\begin{aligned} G_0 &= \|\nabla\widetilde{\psi}\|^2 + \|\nabla\widetilde{\phi}\|^2 + \|\widetilde{B}_z\|^2 + \|\widetilde{V}_z\|^2 \\ &+ 2a^{-1}(\widetilde{\psi}, \widetilde{B}_z) - 2b^{-1}(\widetilde{\psi} + \widetilde{V}_z, \widetilde{B}_z - \Delta\widetilde{\phi}) \end{aligned} \tag{4.65}$$

is a constant of motion of the perturbations $\widetilde{\psi}$, $\widetilde{\phi}$, $\widetilde{B}_z$ and $\widetilde{V}_z$, which satisfy the nonlinear equations (4.61) - (4.64) or their linearized equations.

In what follows, we restrict ourselves to considering the perturbations that satisfy the linearized equations of (4.61) - (4.64). Owing to the Beltrami condition (4.58), the linearized equations can be expressed as

$$\partial_t(-\Delta\widetilde{\phi}) + \{\phi_0, -\Delta\widetilde{\phi} - (b^2-1)\widetilde{\phi} + (b-a^{-1})\widetilde{\psi}\}$$
$$+\{\psi_0, \Delta\widetilde{\psi} + (b-a^{-1})\widetilde{\phi} + (a^{-2}-1)\widetilde{\psi}\} = 0, \tag{4.66}$$
$$\partial_t\widetilde{V}_z + \{\phi_0, \widetilde{V}_z - b\widetilde{\phi} + \widetilde{\psi}\}$$
$$+\{\psi_0, \widetilde{\phi} - a^{-1}\widetilde{\psi} - \widetilde{B}_z\} = 0, \tag{4.67}$$
$$\partial_t\widetilde{\psi} + \{\psi_0, -\widetilde{\phi} + a^{-1}\widetilde{\psi} + \widetilde{B}_z\} = 0, \tag{4.68}$$
$$\partial_t\widetilde{B}_z + \{\phi_0, \widetilde{B}_z - \widetilde{\phi} + a^{-1}\widetilde{\psi}\}$$
$$+\{\psi_0, a^{-1}\widetilde{\phi} - a^{-2}\widetilde{\psi} - \widetilde{V}_z - \Delta\widetilde{\psi}\} = 0. \tag{4.69}$$

In two special cases, we can find a constant of motion that is in a coercive form, and can then derive a stability condition.

4.3.3 Longitudinal flow system with a spiral magnetic field

The first case where we can find constants of motion besides G_0 is a longitudinal flow system with a spiral magnetic field, that is, $\phi_0 = 0$, (we have discussed this kind of equilibrium in Sec. 3.2.3). From (4.58), the equilibrium may be written as $a^{-1} = b$, $C_B = bC_V$ since $\Delta\phi_0 = 0$, and

$$\begin{cases} B_{z0} = -b\psi_0 + bC_V, \\ V_{z0} = -\psi_0 + C_V, \\ \Delta\psi_0 = (1-b^2)(\psi_0 - C_V). \end{cases} \tag{4.70}$$

The two eigenvalues of are written as $\lambda_\pm = \pm(b^2-1)^{1/2}$, see (2.29).

To such an equilibrium, the set of dynamic equations (4.66) - (4.69) is reduced to

$$\partial_t(-\Delta\widetilde{\phi}) + \{\psi_0, \Delta\widetilde{\psi} + (b^2-1)\widetilde{\psi}\} = 0, \tag{4.71}$$

$$\partial_t \widetilde{V}_z + \{\psi_0, \widetilde{\phi} - b\widetilde{\psi} - \widetilde{B}_z\} = 0, \tag{4.72}$$

$$\partial_t \widetilde{\psi} + \{\psi_0, -\widetilde{\phi} + b\widetilde{\psi} + \widetilde{B}_z\} = 0, \tag{4.73}$$

$$\partial_t \widetilde{B}_z + \{\psi_0, b\widetilde{\phi} - b^2\widetilde{\psi} - \widetilde{V}_z - \Delta\widetilde{\psi}\} = 0. \tag{4.74}$$

Adding (4.72) and (4.73) leads to

$$\partial_t(\widetilde{\psi} + \widetilde{V}_z) = 0. \tag{4.75}$$

Hence, we have the following constants;

$$G_1 = \|\widetilde{\psi} + \widetilde{V}_z\|^2, \tag{4.76}$$

$$G_2 = \|\nabla(\widetilde{\psi} + \widetilde{V}_z)\|^2. \tag{4.77}$$

On the other hand, using G_0 of (4.65) and

$$
\begin{aligned}
&2(\widetilde{\psi}, \widetilde{B}_z) \leq \frac{1}{\alpha_1}\|\widetilde{\psi}\|^2 + \alpha_1\|\widetilde{B}_z\|^2 \quad (\forall\alpha_1 > 0),\\
&2(\widetilde{\psi} + \widetilde{V}_z, \widetilde{B}_z) \leq \frac{1}{\alpha_2}\|\widetilde{\psi} + \widetilde{V}_z\|^2 + \alpha_2\|\widetilde{B}_z\|^2 \quad (\forall\alpha_2 > 0),\\
&2(\widetilde{\psi} + \widetilde{V}_z, -\Delta\widetilde{\phi}) = 2\left(\nabla(\widetilde{\psi} + \widetilde{V}_z), \nabla\widetilde{\phi}\right)\\
&\qquad \leq \frac{1}{\alpha_3}\|\nabla(\widetilde{\psi} + \widetilde{V}_z)\|^2 + \alpha_3\|\nabla\widetilde{\phi}\|^2 \quad (\forall\alpha_3 > 0),
\end{aligned}
$$

we observe

$$
\begin{aligned}
G_0 \ \geq\ & \|\nabla\widetilde{\psi}\|^2 + \|\nabla\widetilde{\phi}\|^2 + \|\widetilde{B}_z\|^2 + \|\widetilde{V}_z\|^2\\
& -|b|\left(\frac{1}{\alpha_1}\|\widetilde{\psi}\|^2 + \alpha_1\|\widetilde{B}_z\|^2\right)\\
& -\frac{1}{|b|}\left(\frac{1}{\alpha_2}\|\widetilde{\psi} + \widetilde{V}_z\|^2 + \alpha_2\|\widetilde{B}_z\|^2\right)\\
& -\frac{1}{|b|}\left(\frac{1}{\alpha_3}\|\nabla(\widetilde{\psi} + \widetilde{V}_z)\|^2 + \alpha_3\|\nabla\widetilde{\phi}\|^2\right).
\end{aligned} \tag{4.78}
$$

In a bounded domain, the following coerciveness condition (Poincaré type inequality) is satisfied,

$$
\|\nabla\widetilde{\psi}\|^2 \geq \mu^2\|\widetilde{\psi}\|^2, \tag{4.79}
$$

where μ^2 is the smallest eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition. Using (4.76), (4.77) and (4.79), we can rearrange (4.78) to

$$
\begin{aligned}
&G_0 + \frac{1}{\alpha_2|b|}G_1 + \frac{1}{\alpha_3|b|}G_2 \\
&\geq \left(1 - \frac{|b|}{\alpha_1\mu^2}\right)\|\nabla\widetilde{\psi}\|^2 + \left(1 - \frac{\alpha_3}{|b|}\right)\|\nabla\widetilde{\phi}\|^2 \\
&\quad + \left(1 - \alpha_1|b| - \frac{\alpha_2}{|b|}\right)\|\widetilde{B}_z\|^2 + \|\widetilde{V}_z\|^2. \qquad (4.80)
\end{aligned}
$$

If all the coefficients of the right-hand side of (4.80) are positive, we obtain the bounds for the energy associated with the magnetic as well as the velocity fluctuations. This is because G_0, G_1 and G_2 are constants determined by the initial condition of the perturbations, i.e., the left-hand side of (4.80) works as a Lyapunov function. The condition that the first coefficient of right-hand side must be positive demands that we choose $\alpha_1 > |b|/\mu^2$, and then the third one gives

$$
1 > \alpha_1|b| + \frac{\alpha_2}{|b|} > \frac{b^2}{\mu^2} + \frac{\alpha_2}{|b|} > \frac{b^2}{\mu^2}, \qquad (4.81)
$$

which also demands that we choose $\alpha_2 < (1 - b^2/\mu^2)|b|$. As for the coefficient of the second term, we can set it to positive by choosing $\alpha_3 < |b|$. Finally, due to (4.81), the sufficient condition for the Lyapunov stability can be written as

$$
\lambda^2 = b^2 - 1 < \mu^2 - 1, \qquad (4.82)
$$

where $\lambda^2 = \lambda_\pm^2 = b^2 - 1$, which is the eigenvalue of the double Beltrami equilibrium (4.70) and characterizes the twist of the field, see (2.29). The condition (4.82) gives the upper limit of λ^2 of a stable equilibrium, relating to μ^2 that is the smallest eigenvalue of the Laplacian.

4.3.4 Longitudinal magnetic field system with a spiral flow

Next, we will give another equilibrium field whose stability can be analyzed by invoking constants of motion. We suppose a longitudinal magnetic field system with a spiral flow, that is, $\psi_0 = 0$. From (4.58), the equilibrium can be written as $a^{-1} = b$, $bC_B = C_V$ since $\Delta\psi_0 = 0$, and

$$\begin{cases} B_{z0} = \phi_0 + C_B, \\ V_{z0} = b\phi_0 + bC_B, \\ \Delta\phi_0 = (1 - b^2)(\phi_0 + C_B). \end{cases} \tag{4.83}$$

The two eigenvalues are written as $\lambda_\pm = \pm(b^2 - 1)^{1/2}$.

This equilibrium reduces the dynamic equations of perturbations (4.66) -

(4.69) to

$$\partial_t(-\Delta\widetilde{\phi}) + \{\phi_0, -\Delta\widetilde{\phi} - (b^2-1)\widetilde{\phi}\} = 0, \tag{4.84}$$

$$\partial_t\widetilde{V}_z + \{\phi_0, \widetilde{V}_z - b\widetilde{\phi} + \widetilde{\psi}\} = 0, \tag{4.85}$$

$$\partial_t\widetilde{\psi} = 0, \tag{4.86}$$

$$\partial_t\widetilde{B}_z + \{\phi_0, \widetilde{B}_z - \widetilde{\phi} + b\widetilde{\psi}\} = 0. \tag{4.87}$$

We easily verify that

$$\begin{aligned} G_3 &= (-\Delta\widetilde{\phi}, -\Delta\widetilde{\phi} - \lambda^2\widetilde{\phi}) \\ &= \| -\Delta\widetilde{\phi}\|^2 - \lambda^2\|\nabla\widetilde{\phi}\|^2 \end{aligned} \tag{4.88}$$

is a constant of motion associated with (4.84), where $\lambda^2 = \lambda_\pm^2 = b^2 - 1$. We can find this kind of constant in two-dimensional vortex dynamics, see Sec. 4.1.2. In a bounded domain, we have the inequality

$$\| -\Delta\widetilde{\phi}\|^2 \geq \mu^2\|\nabla\widetilde{\phi}\|^2 \tag{4.89}$$

where μ^2 is the smallest eigenvalue of the Laplacian $-\Delta$ with the Dirichlet boundary condition. Thus we have

$$G_3 \geq \left(1 - \frac{\lambda^2}{\mu^2}\right) \| -\Delta\widetilde{\phi}\|^2, \tag{4.90}$$

or

$$G_3 \geq \left(\mu^2 - \lambda^2\right) \|\nabla\widetilde{\phi}\|^2, \tag{4.91}$$

implying that, when $\lambda^2 < \mu^2$, the enstrophy $\| -\Delta\widetilde{\phi}\|^2$ and the energy $\|\nabla\widetilde{\phi}\|^2$ remain bounded by G_3, which is a constant determined by the initial condition of $\widetilde{\phi}$. From (4.86), we have another constant

$$G_4 = \|\widetilde{\psi}\|^2. \tag{4.92}$$

On the other hand, G_0 of (4.65) leads to the following inequality

$$\begin{aligned}
G_0 &= \|\nabla\widetilde{\psi}\|^2 + \|\nabla\widetilde{\phi}\|^2 + \|\widetilde{B}_z\|^2 + \|\widetilde{V}_z\|^2 \\
&\quad +2\left(b - \frac{1}{b}\right)\left(\widetilde{\psi}, \widetilde{B}_z\right) \\
&\quad -\frac{2}{b}\left[\left(\widetilde{\psi}, -\Delta\widetilde{\phi}\right) + \left(\widetilde{V}_z, \widetilde{B}_z\right) + \left(\widetilde{V}_z, -\Delta\widetilde{\phi}\right)\right] \\
&\geq \|\nabla\widetilde{\psi}\|^2 + \|\nabla\widetilde{\phi}\|^2 + \|\widetilde{B}_z\|^2 + \|\widetilde{V}_z\|^2 \\
&\quad -\left|b - \frac{1}{b}\right|\left[\frac{\|\widetilde{\psi}\|^2}{\beta_1} + \beta_1\|\widetilde{B}_z\|^2\right] \\
&\quad -\frac{1}{|b|}\left[\frac{\|\widetilde{\psi}\|^2}{\beta_2} + \beta_2\| -\Delta\widetilde{\phi}\|^2 + \frac{\|\widetilde{V}_z\|^2}{\beta_3} + \beta_3\|\widetilde{B}_z\|^2\right. \\
&\qquad \left. +\frac{\|\widetilde{V}_z\|^2}{\beta_4} + \beta_4\| -\Delta\widetilde{\phi}\|^2\right],
\end{aligned} \tag{4.93}$$

where β_1, β_2, β_3 and β_4 are arbitrary positive constants. Using (4.90) and

(4.92), the inequality (4.93) is rearranged to

$$
\begin{aligned}
G_0 &+ \frac{\beta_2+\beta_4}{\left(1-\frac{\lambda^2}{\mu^2}\right)}\frac{G_3}{|b|} + \left[\frac{\left|b-\frac{1}{b}\right|}{\beta_1} + \frac{1}{|b|\beta_2}\right] G_4 \\
&\geq \|\nabla\widetilde{\psi}\|^2 + \|\nabla\widetilde{\phi}\|^2 + \left(1-\left|b-\frac{1}{b}\right|\beta_1 - \frac{\beta_3}{|b|}\right)\|\widetilde{B}_z\|^2 \\
&\quad + \left[1-\frac{1}{|b|}\left(\frac{1}{\beta_3}+\frac{1}{\beta_4}\right)\right]\|\widetilde{V}_z\|^2. \qquad (4.94)
\end{aligned}
$$

If $\lambda^2 < \mu^2$ and the coefficients of both $\|\widetilde{B}_z\|^2$ and $\|\widetilde{V}_z\|^2$ in the right-hand side are positive, (4.94) gives a bound for the energy associated with fluctuations. The positivity of the coefficient of $\|\widetilde{V}_z\|^2$ demands

$$
|b| > \frac{1}{\beta_3} + \frac{1}{\beta_4} > \frac{1}{\beta_3}. \qquad (4.95)
$$

Then the positivity of the coefficient of $\|\widetilde{B}_z\|^2$ requires

$$
0 < 1 - \left|b-\frac{1}{b}\right|\beta_1 - \frac{1}{b^2} < 1 - \frac{1}{b^2}. \qquad (4.96)
$$

We can thus conclude that

$$
0 < \lambda^2 = b^2 - 1 < \mu^2 \qquad (4.97)
$$

is the sufficient condition for the Lyapunov stability. The condition (4.97) demands that λ^2 of a stable state must not exceed μ^2 that is the smallest eigenvalue of the Laplacian.

4.3.5 Discussion

We have found the Lyapunov function that yields a bound of all possible fluctuations for two special configurations of the double Beltrami field. In this analysis, the sufficient condition of stability is obtained by relating the Beltrami parameters $\lambda_\pm$ to the smallest eigenvalue of the Laplacian in the bounded domain.

Although we have the constant of motion G of (4.55) for the general double Beltrami field, this constant is not a coercive form in the energy norm. The coerciveness demands that G must be a convex form, which is related to the index of the Hessian, as mentioned in Sec. 4.2. Therefore we cannot apply the standard (Arnold's) method [54, 55, 56] for stability based on the second variation of G adopted by Holm *et al* [62]. The important point here is that we can find other constants that are of the coercive (convex) form. In our case, these are the enstrophy order constants, i.e., G_2 of (4.77) or G_3 of (4.88), since the coerciveness is measured by the highest order of derivatives included in the constants. By invoking G_2 or G_3, we can apply the coerciveness relation (4.79) or (4.89) to derive the bound for the energy of the perturbations.

These constants of motion, however, depend strongly on the geometry. As we showed in Sec. 4.1.2, the enstrophy is an important constant in a two-dimensional incompressible (neutral) flow. However, in a three-dimensional flow, the convective derivative destroys the conservation of the enstrophy because of a vortex-stretching effect. Moreover, the two-fluid MHD has two characteristics – the velocity field and the magnetic field – that make it more complicated and difficult to find a constant (integral) of motion of perturbations. We can find the Lyapunov function to the equilibrium that has only one characteristic curve ψ_0 (Sec. 4.3.3) or ϕ_0 (Sec. 4.3.4). However, it is still an open question whether we can find such a kind of bound for general two-fluid MHD equilibrium.

Chapter 5

Hall current and Alfvén wave

5.1 Alfvén wave in MHD

In its simplest manifestation, the Alfvén wave, one of the most important waves in magnetohydrodynamics (MHD), propagates along the magnetic field, and its frequency is determined entirely by the wave number in the direction parallel to the magnetic field (independent of the perpendicular wave numbers) [31]. The spectral structure of the Alfvén wave has a close relationship with the geometric structure of the magnetic field lines. The fact that the characteristics, which are the direction of the wave energy propagation, are identical to the ambient magnetic field lines implies that the Alfvén wave equation has degenerate characteristics. Indeed, in an inhomogeneous MHD plasma, the Alfvén wave is the solution of a second-order singular differential equation implying a continuous spectrum [28, 29, 30].

It is expected, however, that, a more realistic treatment of the plasma (inclusion of non-ideal MHD effect, for example) should yield a nonsingular system with discrete eigenmodes of the Alfvén wave. Such qualitative changes in the spectrum are relevant to the understanding of many plasma phenomena [32]. Adding finite resistivity to the ideal MHD allows the development of the tearing mode instability represented by an imaginary frequency point spectrum occurring through a singular perturbation of zero frequency spectrum (the edge of the Alfvén continuous spectrum) [33]. Within the framework of an ideal plasma, there exist mechanisms which cure the pathology inherent in the ideal MHD and lead to a nonsingular equation yielding discrete spectra with well-defined non-infinite perturbations. Kinetic effects, in particular, the electron inertia have been invoked in the past to resolve the ideal MHD singularity [34].

The ability of electron inertia to resolve the singularity seems to suggest that any physical effect that distinguishes between the electron and the ion motions (ideal MHD does not) may have the potential to generate a discrete spectrum. The simplest non-dissipative two-fluid model in which the electron and ion velocities differ from the standard $\boldsymbol{E} \times \boldsymbol{B}$ drift is the Hall MHD (discussed in Sec. 2.1). It is of considerable importance, therefore, to investigate the structure of Alfvén waves in Hall MHD [67]. In particular one would like to know if the Alfvén wave in inhomogeneous Hall MHD sheds its singular character, and if yes under what conditions does the singularity disappear.

Settling the question of "the continuous spectrum" in Hall MHD has consequences beyond the immediate aim stated above. Since the structure of the Alfvén wave in an inhomogeneous medium is determined by the mechanism which removes the singularity, it is important to ascertain the dominant mechanism for a given plasma. We must, for example, assess the relative merits of the electron inertia and the Hall term (electron inertia is zero in Hall MHD) in singularity resolution and only then we will be able to predict the nature of Alfvén eigenmodes (for example, the mode width). We shall later show that the Hall term will generally dominate in standard laboratory plasmas and thus will be the primary determinant of the mode structure.

This chapter is devoted to an analysis of Alfvénic perturbations in Hall MHD. Since the Hall term contains a higher order spatial derivative than other terms in ideal MHD, it may be expected to remove the Alfvén singularity. We shall soon see, however, that the Hall term alone is not enough; in the zero beta limit (no sound wave), it simply shifts the singularity without resolving it. It is joint effort of the combination of the Hall current with finite beta which leads to a nonsingular equation yielding a discrete spectrum.

We derive a homogeneous plasma dispersion relation in Sec. 5.2. The homogeneous plasma dispersion relation is rather revealing; it clearly indicates what we stated in the preceding paragraph that it is the coupling of sound wave and the hall current that is needed to provide higher order dispersion necessary for singularity removal.

We then graduate to the derivation of the mode equation for inhomogeneous Hall MHD in Sec. 5.3. From the mode equation we systematically draw inferences that we stated earlier.

(A) Though the Hall term contributes a higher order derivative, without pressure it cannot remove the singularity. In a zero-pressure plasma, it leads to the following changes: (i) wave frequency shift, (ii) polarization change from linear to elliptic, and (iii) the possibility of global Alfvén eigenmodes (GAE) with discrete spectrum provided the Hall effect is strong enough.

(B) The Hall term couples with the finite pressure (sound wave) term to produce the higher order (fourth order) derivative that removes the singularity in the Alfvén wave equation.

5.2 Dispersion relation of Hall MHD

In order to take account of the coupling of Hall current and sound wave, we consider the Hall MHD equation with compressible flow governed by

$$n\left[\partial_t \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right] = (\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \nabla p, \tag{5.1}$$

$$\partial_t \boldsymbol{B} = \nabla\times\left[\left(\boldsymbol{V} - \varepsilon\frac{\nabla\times\boldsymbol{B}}{n}\right)\times\boldsymbol{B}\right], \tag{5.2}$$

$$\partial_t n + \nabla\cdot(n\boldsymbol{V}) = 0, \tag{5.3}$$

with the adiabatic equation of state $d\left(p\,n^{-\gamma}\right)/dt = 0$, where γ is the adiabatic constant. Since the Hall-term $\varepsilon(\nabla\times\boldsymbol{B}/n)\times\boldsymbol{B}$ in (5.2), where the scaling coefficient ε is a measure of the ion skin depth (see Sec. 2.1), contains higher order derivatives compared to the other terms in the ideal MHD equation ($\varepsilon = 0$), one expects at the outset that this higher derivative modification will cure the Alfvén singularity.

In order to gain some insight into the dispersive character of the Hall MHD waves, we begin with deriving the flow-less homogeneous dispersion relation. We linearize (5.1) - (5.3) about the equilibrium $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$, $\boldsymbol{V}_0 = 0$, and

$n_0 = 1$, as

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{v}, \\
\boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{b}, \qquad (5.4) \\
n &= 1 + \widetilde{n},
\end{aligned}$$

with perturbations proportional to $\exp[-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})]$. The perturbed equation of state $\nabla\widetilde{p} = C_s^2 \nabla\widetilde{n}$ ($C_s = \sqrt{\gamma p/n}$ is the sound speed), and the continuity equation (5.3) into $\widetilde{n} = (\boldsymbol{k}\cdot\boldsymbol{v})/\omega$ yield the system

$$\begin{aligned}
-\omega\boldsymbol{v} &= B_0(\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{e}_z - \frac{C_s^2}{\omega}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k} \\
&= B_0(k_z\boldsymbol{b} - b_z\boldsymbol{k}) - \frac{C_s^2}{\omega}(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{k}, \qquad (5.5) \\
-\omega\boldsymbol{b} &= B_0\boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{e}_z) - i\varepsilon B_0\boldsymbol{k}\times[(\boldsymbol{k}\times\boldsymbol{b})\times\boldsymbol{e}_z] \\
&= B_0[k_z\boldsymbol{v} - (\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{e}_z] - i\varepsilon B_0 k_z(\boldsymbol{k}\times\boldsymbol{b}). \qquad (5.6)
\end{aligned}$$

Remembering $\boldsymbol{k}\cdot\boldsymbol{b} = 0$ (due to $\nabla\cdot\boldsymbol{B} = 0$), the inner product of $\boldsymbol{k}$ and (5.5) leads to

$$\boldsymbol{k}\cdot\boldsymbol{v} = \frac{B_0 k^2 \omega b_z}{\omega^2 - C_s^2 k^2}, \qquad (5.7)$$

and then, combining the z-components of (5.5) and (5.6), we get

$$\begin{aligned}
-\omega b_z &= B_0\left(\frac{C_s^2 k_z^2}{\omega^2} - 1\right)(\boldsymbol{k}\cdot\boldsymbol{v}) - i\varepsilon B_0 k_z j_z \\
&= -B_0^2 k^2 b_z \frac{\omega^2 - C_s^2 k_z^2}{\omega(\omega^2 - C_s^2 k^2)} - i\varepsilon B_0 k_z j_z, \qquad (5.8)
\end{aligned}$$

where $j_z = (\boldsymbol{k} \times \boldsymbol{b}) \cdot \boldsymbol{e}_z$. Further manipulation of (5.6), and (5.5) gives

$$-\omega j_z = B_0 k_z (\boldsymbol{k} \times \boldsymbol{v}) \cdot \boldsymbol{e}_z + i\varepsilon B_0 k_z k^2 b_z, \tag{5.9}$$

$$-\omega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \boldsymbol{e}_z = B_0 k_z j_z, \tag{5.10}$$

from which j_z can be expressed as

$$j_z = -i\varepsilon \frac{B_0 k_z k^2 \omega b_z}{\omega^2 - B_0^2 k_z^2}. \tag{5.11}$$

Substituting (5.11) in (5.8), we obtain the dispersion relation of Hall MHD as

$$\begin{aligned} \left(\omega^2 - B_0^2 k_z^2\right) \left[\omega^4 - \left(C_s^2 + B_0^2\right) k^2 \omega^2 + C_s^2 B_0^2 k_z^2 k^2\right] \\ = \varepsilon^2 B_0^2 k_z^2 k^2 \omega^2 \left(\omega^2 - C_s^2 k^2\right). \end{aligned} \tag{5.12}$$

Clearly, setting $\varepsilon = 0$ reduces (5.12) to the MHD dispersion relation, see (3.8).

The Hall MHD dispersion relation differs from the standard MHD relation by the ε-dependent term on the right hand side of (5.12). Notice, however, that the existence of the higher order dispersion (proportional to k^4) requires that ε^2 and C_s^2 be simultaneously nonzero. The vanishing of either changes, at best, the coefficient of the k^2 term. The translation to the inhomogeneous language (to be discussed soon) is that the fourth order derivative term needed to remove the singularity associated with the second order differential equation is possible only when the Hall effect is augmented by finite beta. The Hall term alone simply shifts the zero of the coefficient of the second order term.

5.3 Alfvén wave in inhomogeneous Hall MHD

The calculation to investigate the conditions under which the Alfvén singularity in an inhomogeneous plasma is resolved, constitutes the main thrust of this work. We deal with the simplest nontrivial problem: an inhomogeneous one-dimensional slab (Cartesian) with the ambient magnetic field in the form

$$\boldsymbol{B}_0 = \frac{\boldsymbol{e}_z + f(x)\,\boldsymbol{e}_y}{\sqrt{1+f^2(x)}}. \tag{5.13}$$

The proposed $\boldsymbol{B}_0$ satisfies

$$\nabla \times \boldsymbol{B}_0 = h\boldsymbol{B}_0, \qquad h(x) = \frac{f'}{1+f^2}, \tag{5.14}$$

where $f' = d\,f(x)/d\,x$. If we choose $f \ll 1$, $\boldsymbol{B}_0$ has much stronger z-field than y-field like the tokamak magnetic field; on the other hand if we choose $f = \tan(\lambda x)$ with constant λ, $\boldsymbol{B}_0$ expresses the Cartesian version of a Taylor relaxed (single Beltrami) magnetic field of RFP, $\nabla \times \boldsymbol{B}_0 = \lambda \boldsymbol{B}_0$ [18, 19]. The equilibrium magnetic field (5.14) allows us to define the following orthogonal unit vectors:

$$\boldsymbol{e}_x, \quad \boldsymbol{e}_\parallel = \boldsymbol{B}_0 = \frac{\boldsymbol{e}_z + f\boldsymbol{e}_y}{\sqrt{1+f^2}}, \quad \boldsymbol{e}_\perp = \frac{\boldsymbol{e}_y - f\boldsymbol{e}_z}{\sqrt{1+f^2}}, \tag{5.15}$$

satisfying $\boldsymbol{e}_\perp \times \boldsymbol{e}_\parallel = \boldsymbol{e}_x$ and $\nabla \times \boldsymbol{e}_{\parallel(\perp)} = h\ \boldsymbol{e}_{\parallel(\perp)}$. To derive the wave equation in the inhomogeneous variable x, we first linearize (5.1) - (5.3) around the equilibrium, and then Fourier expand the perturbations in the ignorable directions: $\widetilde{q} = q(x)\exp[-i(\omega t - k_y y - k_z z)]$. The explicit unit vectors (5.15) help to define the wave numbers

$$k_\parallel = \frac{k_z + f k_y}{\sqrt{1+f^2}}, \quad k_\perp = \frac{k_y - f k_z}{\sqrt{1+f^2}}, \tag{5.16}$$

obeying

$$\begin{aligned} & k_\perp^2 + k_\parallel^2 = k_y^2 + k_z^2 = \text{constant}, \\ & \tfrac{d}{dx} k_\perp = -h k_\parallel, \quad \tfrac{d}{dx} k_\parallel = h k_\perp. \end{aligned} \tag{5.17}$$

Following standard methods, linearized equations (5.1) and (5.2) can be written as

$$-i\omega \boldsymbol{v} = (\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{b} + \frac{iC_s^2}{\omega} \nabla(\nabla \cdot \boldsymbol{v}), \tag{5.18}$$

$$\begin{aligned} -i\omega \boldsymbol{b} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0) - \varepsilon \nabla \times [(\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{b}] \\ &= \nabla \times [\boldsymbol{v} \times \boldsymbol{B}_0 + i\varepsilon\omega \boldsymbol{v}] \equiv \nabla \times \boldsymbol{g}. \end{aligned} \tag{5.19}$$

To arrive at the final form (5.19), (5.18) was used, and a new variable $\boldsymbol{g} \equiv \boldsymbol{v} \times \boldsymbol{B}_0 + i\varepsilon\omega \boldsymbol{v}$ was introduced. Combining (5.18) and (5.19), we obtain

$$-\omega^2 \boldsymbol{v} = (\nabla \times \nabla \times \boldsymbol{g} - h \nabla \times \boldsymbol{g}) \times \boldsymbol{e}_\parallel + C_s^2 \nabla(\nabla \cdot \boldsymbol{v}). \tag{5.20}$$

From the definition of $\boldsymbol{g}$, we have

$$\begin{aligned} g_\parallel &= i\varepsilon\omega v_\parallel, \\ g_x &= v_\perp + i\varepsilon\omega v_x, \\ g_\perp &= -v_x + i\varepsilon\omega v_\perp, \end{aligned} \tag{5.21}$$

and the resulting reciprocal relations

$$\begin{aligned} v_x &= \frac{1}{1-\varepsilon^2\omega^2}\left[-g_\perp + i\varepsilon\omega g_x\right], \\ v_\perp &= \frac{1}{1-\varepsilon^2\omega^2}\left[g_x + i\varepsilon\omega g_\perp\right]. \end{aligned} \tag{5.22}$$

For the proposed geometry it is straightforward to evaluate the curl operator. In particular, we derive

$$\begin{aligned} \nabla\times\boldsymbol{g} &= i(k_\perp g_\parallel - k_\parallel g_\perp)\boldsymbol{e}_x + \left(ik_\parallel g_x - \frac{d}{dx}g_\parallel + hg_\perp\right)\boldsymbol{e}_\perp \\ &\quad + \left(\frac{d}{dx}g_\perp - ik_\perp g_x + hg_\parallel\right)\boldsymbol{e}_\parallel. \end{aligned} \tag{5.23}$$

Breaking (5.20) into components and using various relations that we have displayed, we arrive at the general system describing waves in an inhomogeneous Hall MHD plasma,

$$\begin{aligned} \left(\Omega^2 - k_\parallel^2 + \frac{d^2}{dx^2}\right) g_\perp &= i\left[\frac{d}{dx}(k_\perp g_x) + \varepsilon\omega\Omega^2 g_x\right] \\ &\quad + \frac{d}{dx}\left[\left(\frac{\omega}{\varepsilon k_\parallel} - h\right) g_\parallel\right] - k_\parallel k_\perp g_\parallel, \end{aligned} \tag{5.24}$$

$$\begin{aligned} \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right) g_x &= i\left(k_\perp \frac{d}{dx} - \varepsilon\omega\Omega^2\right) g_\perp \\ &\quad + i\left(k_\parallel \frac{d}{dx} - k_\perp \frac{\omega}{\varepsilon k_\parallel}\right) g_\parallel, \end{aligned} \tag{5.25}$$

and

$$g_{\|} = \frac{\varepsilon^2 C_s^2 \Omega^2 k_{\|}}{\omega^2 - k_{\|}^2 C_s^2} \left[i \left(\frac{d}{dx} + \frac{k_\perp}{\varepsilon\omega} \right) g_x - \left(\frac{1}{\varepsilon\omega}\frac{d}{dx} + k_\perp \right) g_\perp \right], \tag{5.26}$$

where $\Omega^2 \equiv \omega^2/(1-\varepsilon^2\omega^2)$.

5.3.1 Zero-pressure case

We shall proceed in incremental steps to illustrate the nature of Hall MHD modes. First we consider the zero-pressure $C_s^2 = 0$ limit for which the parallel components of the perturbed velocity and of the auxiliary variable $\boldsymbol{g}$ vanish ($g_{\|} = 0$, $v_{\|} = 0$); this follows from (5.20) and (5.26). Under this condition, (5.24) and (5.25) are easily combined to yield

$$\frac{d}{dx}\left(\Omega^2 - k_{\|}^2\right)\frac{d}{dx}g_\perp + \left(\Omega^2 - k_{\|}^2 - k_\perp^2\right)\left(\Omega^2 - k_{\|}^2\right) g_\perp = \varepsilon\omega\Omega^2\left(\varepsilon\omega\Omega^2 - k_{\|}h\right) g_\perp, \tag{5.27}$$

where the relation (5.17) has been used. The limit $\varepsilon \to 0$ ($\Omega^2 \to \omega^2$, $g_\perp \to -v_x$) reduces (5.27) to the second-order singular differential equation of ideal MHD. The singularity is at $\omega^2 = k_{\|}^2(x)$ and the general solution is of the Frobenius-type corresponding to the continuous spectrum. The effect of the finite Hall current on the Alfvén wave, therefore, can be summarized as follows: (i) the singularity shifts from $\omega^2 = k_{\|}^2(x)$ to $\Omega^2 = k_{\|}^2(x)$, (ii) the polarization of

Alfvén wave changes from linear to elliptic, which is clearly expressed by $\boldsymbol{g}$; see (5.21), (iii) the extra term on the right-hand side of (5.27) (equivalent to the modifications of the basic MHD equation by the magnetic shear and other effects [34]) has the potential to allow a discrete spectrum of global Alfvén eigenmodes (GAE) below the MHD continuum (Sec. 5.4). However, as indicated by the dispersion relation (5.12), the Hall current without pressure cannot remove the basic singularity; it is simply shifted to $\Omega^2 = k_\parallel^2(x)$.

5.3.2 Finite-pressure case

Going back to the finite pressure Hall MHD system, we formally eliminate g_x in (5.24) - (5.26) to obtain

$$\begin{aligned}
&\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left(\Omega^2 - k_\parallel^2\right)g_\perp - \varepsilon\omega\Omega^2\left(\varepsilon\omega\Omega^2 - k_\parallel h\right)g_\perp \\
&\quad = -\frac{d}{dx}\left(k_\perp k_\parallel \frac{d}{dx}g_\parallel\right) + \frac{d}{dx}\left(k_\perp^2 \frac{\omega}{\varepsilon k_\parallel}g_\parallel\right) - \varepsilon\omega\Omega^2\left(k_\parallel \frac{d}{dx}g_\parallel + k_\perp \frac{\omega}{\varepsilon k_\parallel}g_\parallel\right) \\
&\qquad + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left\{\frac{d}{dx}\left[\left(\frac{\omega}{\varepsilon k_\parallel} - h\right)g_\parallel\right] - k_\parallel k_\perp g_\parallel\right\}. \qquad (5.28)
\end{aligned}$$

The right-hand side of (5.28) shows the finite pressure effects induced by finite $g_\parallel$. The system is rather complicated [and in reality, it has both $g_\perp$, and g_x (through $g_\parallel$)] and a detailed analysis should be done numerically. An approximate analytic calculation, guided by the fact that the fourth order derivative term requires both ε^2 and C_s^2 indicated by the dispersion relation (5.12), is, however, rather instructive. Let us assume that the pressure, though finite, is small (small C_s^2). In this limit the terms on the right-hand side may be ordered small.

The highest order derivative operating on $g_\perp$ comes from the first term of the right-hand side of (5.28). Since this term is the precise object of our attention, we neglect all other terms. In order to find an expression for the highest derivative term, we first substitute

$$g_x \simeq \frac{i}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} k_\perp \frac{d}{dx} g_\perp, \tag{5.29}$$

in (5.26) and estimate

$$g_\parallel \simeq -\varepsilon^2 \delta^2 \frac{k_\perp k_\parallel \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^2}{dx^2} g_\perp, \tag{5.30}$$

where $\delta^2 \equiv C_s^2/(\omega^2 - k_\parallel^2 C_s^2) \simeq C_s^2/\omega^2$. Finally we approximate the fourth order derivative term

$$-\frac{d}{dx}\left(k_\perp k_\parallel \frac{d}{dx} g_\parallel\right) \simeq \varepsilon^2 \delta^2 \frac{k_\perp^2 k_\parallel^2 \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^4}{dx^4} g_\perp, \tag{5.31}$$

and substitute it in (5.28) to derive

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx} g_\perp + \left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)\left(\Omega^2 - k_\parallel^2\right) g_\perp - \varepsilon\omega\Omega^2\left(\varepsilon\omega\Omega^2 - k_\parallel h\right) g_\perp$$
$$\simeq \varepsilon^2 \delta^2 \frac{k_\perp^2 k_\parallel^2 \Omega^2}{\left(\Omega^2 - k_\parallel^2 - k_\perp^2\right)} \frac{d^4}{dx^4} g_\perp, \tag{5.32}$$

simplified as

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp - k_\perp^2\left(\Omega^2 - k_\parallel^2\right)g_\perp - \varepsilon\omega\Omega^2\left(\varepsilon\omega\Omega^2 - k_\parallel h\right)g_\perp$$
$$\simeq -\varepsilon^2\delta^2 k_\parallel^2\Omega^2\frac{d^4}{dx^4}g_\perp. \tag{5.33}$$

To arrive at (5.33), the standard assumption $\Omega^2 - k_\parallel^2 \ll k_\perp^2$ has been invoked. It is clear that $\Omega^2 = k_\parallel^2$ is no more a singular point of the differential equation (5.33); the singular solutions, which constitute the MHD continuum, are no longer there in finite pressure Hall MHD. The Hall current coupling with the sound wave, formally represented by a singular perturbation, contributes a fourth order derivative term, and this perturbation converts the continuous Alfvén spectrum into a point spectrum. We will see this modification clearly in the case when $k_\parallel^2$ has a minimum, in the next section.

5.4 Analysis of the mode equation

Following the methodology of Mahajan [34], we will delineate the general features of the eigenvalue problem described by the equation, [compare with (5.33)]

$$\frac{d}{dx}\left(\Omega^2 - k_\parallel^2\right)\frac{d}{dx}g_\perp - k_\perp^2\left(\Omega^2 - k_\parallel^2\right)g_\perp - \alpha g_\perp = -\beta\frac{d^4}{dx^4}g_\perp. \tag{5.34}$$

A crucial and simplifying step in this method is to assume that in the domain of interest, $k_\parallel^2$ has a well-defined minimum at x_0, and the mode is localized around this minimum. Near x_0, we may write (Ω_0^2 is the minimum of $k_\parallel^2$)

$$k_\parallel^2 = \Omega_0^2 + \Delta_x^2 \xi^2, \quad \Omega^2 = \Omega_0^2 + \Delta_x^2 \mu, \tag{5.35}$$

where Δ_x is a small parameter and μ represents the effective new spectrum. Now we replace all other parameters in the system by their values at x_0 converting (5.34) into

$$\frac{d}{d\xi}\left(\mu - \xi^2\right)\frac{d}{d\xi} g_\perp - k_\perp^2 \Delta_x^2 \left(\mu - \xi^2\right) g_\perp - \alpha g_\perp = -\frac{\beta}{\Delta_x^4}\frac{d^4}{d\xi^4} g_\perp, \tag{5.36}$$

where $\alpha, \beta, \mu, \ldots$ are to be treated independent of ξ. Let us normalize the scale as $k_\perp^2 \Delta_x^2 = 1$. For a sufficiently large $k_\perp^2$ such that $\Delta_x^2 \ll \Omega_0^2$, we may stretch the range of ξ to be $(-\infty, \infty)$. The Fourier transform of (5.36) yields the equation obeyed by $g(\zeta) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} g_\perp \exp(-i\xi\zeta) d\xi$,

$$\frac{d}{d\zeta}\left(1 + \zeta^2\right)\frac{d}{d\zeta} g + \mu\left(1 + \zeta^2\right) g + \alpha g - \frac{\beta}{\Delta_x^4}\zeta^4 g = 0. \tag{5.37}$$

We can further transform (5.37) using a new dependent variable $\psi = (1 + \zeta^2)^{1/2} g$ to yield the effective Schrödinger equation

$$-\frac{d^2}{d\zeta^2}\psi + V(\zeta)\psi = \mu\psi, \tag{5.38}$$

with the effective potential

$$V(\zeta) = \frac{1}{(1+\zeta^2)^2} - \frac{\alpha}{1+\zeta^2} + \frac{\beta}{\Delta_x^4}\frac{\zeta^4}{1+\zeta^2}. \tag{5.39}$$

In the limit $\alpha = \beta = 0$ (no-Hall current), we have a pure scattering potential shown in Fig. 5.1 (a) , which yields the continuous spectrum. For $\alpha \neq 0$ but $\beta = 0$ (Hall current without pressure), the continuous spectrum spans $\mu \geq 0$. However this continuum may be augmented by a discrete spectrum of what are known in the literature as GAE (global Alfvén eigenmodes). The GAE exist for $\mu \leq 0$ if the potential has a sufficiently deep negative dip. The depth of the potential well is controlled by $\alpha > 0$, the measure of the strength of the Hall term. It has been rigorously shown that to obtain a point spectrum, $\alpha > 1/4$ is required, see Fig. 5.1 (b).

The situation is qualitatively different when both α and β are nonzero (Hall current coupling with sound wave). Then, for $\beta > 0$, $V(\zeta) \to \zeta^2$ as $|\zeta| \to \infty$, which is shown in Fig. 5.1 (c). Hence the wave is trapped in the wave number space, and the "eigenvalue" μ is quantized; the MHD continuum for $\mu > 0$ has changed over to a discrete spectrum. The width of mode can be estimated by (5.36) as

$$\Delta_x^4 \simeq \frac{\beta}{\alpha}. \tag{5.40}$$

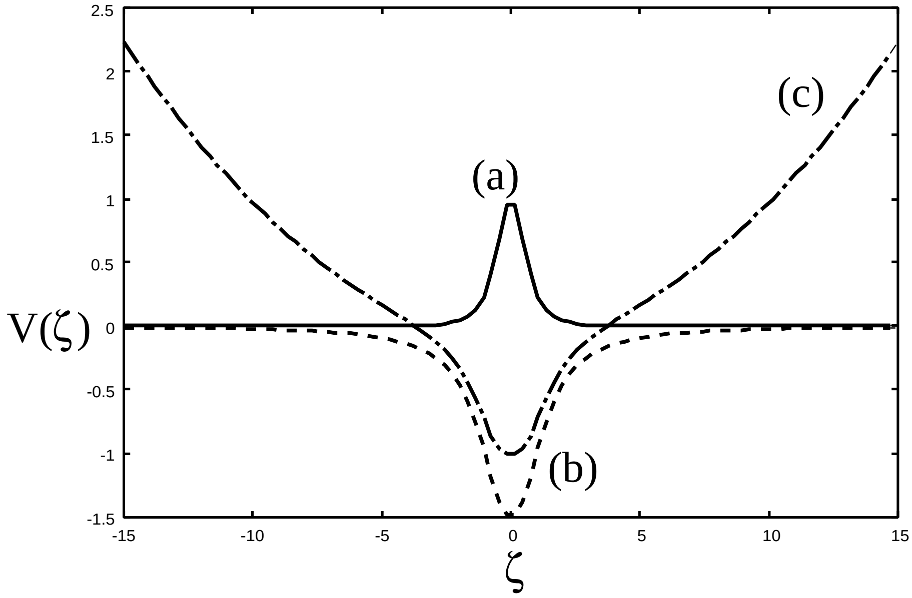

Figure 5.1: $V(\zeta)$ vs ζ. (a) $\alpha = 0$, $\beta = 0$. (b) $\alpha = 2.5$, $\beta = 0$. (c) $\alpha = 2.0$, $\beta = 0.01$. The potential well at large ζ is responsible for the point spectrum.

It is shown that the mode equation (5.33) derived by Hall MHD has a point spectrum, resolving the ideal MHD singularity. It is well known that an entirely similar effect is wrought by finite electron inertia on the ideal MHD spectrum. In a real plasma both these effects are simultaneously present. The question is which one of these will be dominant in a given plasma.

Finally, in order to determine the relative merits of the Hall term and the electron inertia, we calculate the ratio of the widths of the modes (5.40) created when either one of these mechanism is assumed to be dominant. For the Hall dominance scenario, assuming magnetic shear to be small ($h \ll 1$), the width Δ_x of the eigenmode described by (5.32) or (5.33) can be estimated as

$$\Delta_x^4 = \varepsilon\delta^2 \frac{k_\parallel^2}{\omega(\varepsilon\omega\Omega^2 - k_\parallel h)} \simeq \frac{C_s^2}{\omega^4} \simeq \frac{\rho_s^2}{k_\parallel^2}\frac{\omega_{ci}^2}{\omega^2}, \tag{5.41}$$

where $C_s = \rho_s\omega_{ci}$, ρ_s is the ion gyroradius and ω_{ci} is the cyclotron frequency. On the other hand, the width of the eigenmode when electron inertia resolves the singularity is

$$\Delta_{x,e} \simeq (\rho_s a)^{1/2} \simeq \left(\frac{\rho_s}{k_\parallel}\right)^{1/2}, \tag{5.42}$$

where a is the length of plasma domain (radius of cylinder) [34]. The ratio

$$\frac{\Delta_x}{\Delta_{x,e}} \simeq \left(\frac{\omega_{ci}}{\omega}\right)^{1/2} > 1, \tag{5.43}$$

tends to be considerably greater than unity since in the Alfvénic range the cyclotron frequency is considerably greater than the mode frequency. We have stipulated here that $k_\parallel$ is of the order of a^{-1}. This may not be the case in general and $k_\parallel$ could be considerably smaller. But for small parallel wave number the mode frequency tends to be even smaller and the ratio $\Delta_x/\Delta_{x,e}$ remains much larger than unity. It is thus expected, then, that the mode structure will be dominated by the Hall term in the first approximation.

Chapter 6

Summary

We have investigated the equilibrium and the stability of flowing plasmas, and the Alfvén wave spectrum using the two-fluid (Hall) MHD model. The principal term that distinguish the two-fluid model from the standard MHD is the Hall term, which is the highest derivative term in the equation (a singular perturbation of MHD) introducing a intrinsic scale length (the ion skin depth) to the scale-less MHD. Since the Hall MHD degenerates into the MHD without plasma flow, the MHD model may be valid for static phenomena, however it is difficult to say that the MHD is enough to study the dynamics associated with the plasma flow and the small scale (created by the singular perturbation of the Hall term). The two-fluid model is more appropriate to describe magneto-fluid plasma dynamics and resultant complex field. We have studied the Hall effect on the magneto-fluid plasmas relating to plasma flow and small scale structure.

In Chap. 3, we have shown that the Hall term allows the stream function to deviate from magnetic flux function and resolves the singularity that exists in the equilibrium equation of MHD with flow. The equilibrium equation for two-fluid MHD can be cast in a coupled elliptic equations and we have solved the equations as boundary value problem and got a relaxed state that is expressed by the double Beltrami field. Furthermore we have discussed the short scale creation. The short scale may be expressed as a pattern generated by a crossing of the characteristics (the magnetic field $\boldsymbol{B}$ and the flow field $\boldsymbol{V}$), which is also allowed by the Hall term. The emergence of the short scale and its understanding can help us construct the hidden dynamics (not visible at the macro-level), which so effectively determines the nature of the macro structures. This is in sharp contrast to the original ideal MHD model which fails to yield classes of important solutions (equilibrium or slowly evolving states with perpendicular flows) without any artificial symmetry. With the singular perturbation (dispersive effects) contained in Hall MHD, one not only extends the diversity of structures, but also recovers the regularity of solutions.

We have studied the Lyapunov stability of flowing plasmas in Chap. 4. Combining a constant of motion and a coerciveness relation, we have derived a bound for the energy of perturbations yielding a sufficient condition for stability (Lyapunov stability condition). Under appropriate boundary conditions, coerciveness is measured by the highest order of derivatives included in the functional, which is a consequence of a Poincaré-type inequality. We can find the constant of motion that is closely related to the variational principle characterizing the Beltrami equilibrium (**Theorem 1**). Since the constant of motion is a coercive form in MHD, we can regard the constant as a Lyapunov function and obtain the stability condition relating the Beltrami parameters to the eigenvalues of the self-adjoint curl operator. However, for the two-fluid model, the constant of motion related to the variational principle is not a coercive form because of the singular perturbation, which requires an enstrophy order constant. For a special class of two-fluid MHD flows, we can find such kind of constants that are of the coercive (convex) form and conclude the stability condition.

In Chap. 5, we have demonstrated that the combined action of the Hall current and the sound wave (finite beta effects) can resolve the well-known Alfvén wave singularity of an inhomogeneous MHD plasma; neither of these mechanisms, acting alone, is adequate for the job. The Alfvénic fluctuations in finite pressure Hall MHD, therefore, are characterized by a discrete spectrum associated with nonsingular eigenmodes. The Hall mechanism is also likely to dominate the electron inertia route to regularization for most plasmas of interest.

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