Hall effect on turbulence and relaxation process in magneto-fluid plasma

2005 International Sherwood Fusion Theory Conference

Apr 11, 2005 - Apr 13, 2005
Stateline, Nevada, USA (Lake Tahoe)

Shuichi Ohsaki and Zensho Yoshida

IFS, Univ. Texas, Austin, Texas.
A Frontier Sciences, Univ. Tokyo, Japan.
The **Hall effect**, which is written by higher order derivative term, is the **singular perturbation** to the MHD equations.

Microscopic (**ion skin depth order**) scales created by the Hall effect coupled with Alfvénic flows and those influence to the relaxation process of macroscopic fields are studied by numerical simulation.

![Typical time evolution of current governed by Hall MHD](image-url)
Hall term and singular perturbation

Hall-MHD equations

\[
\frac{\partial V}{\partial t} + (V \cdot \nabla)V = (\nabla \times B) \times B - \nabla p + \frac{1}{R_e} \Delta V, \tag{1}
\]

\[
\frac{\partial B}{\partial t} = \nabla \times [(V - \epsilon \nabla \times B) \times B] + \frac{1}{R_m} \Delta B, \tag{2}
\]

\[
\nabla \cdot V = 0. \tag{3}
\]

Scaling coefficient \( \epsilon = l_i/L_0 \) is a measure of the ion skin depth

\[ l_i = c/\omega_{pi} = V_A/\omega_{ci} = \sqrt{M/\mu_0 n e^2}. \]

The only addition to the standard MHD is the Hall current term \( \epsilon (\nabla \times B) \times B \) in (2).

Mathematically, \( \epsilon \) is a singular perturbation parameter, since it multiplies the highest derivative term (in the ideal limit \( 1/R_e = 1/R_m = 0 \)).
Scale hierarchy created in plasma flow


Analysis of equilibrium of Hall MHD

↓

Hall effect on equilibrium

- Crossing $B$ and $V$.
  (perpendicular flow is allowed by the Hall effect.)
- Small scale (ion skin depth order) structures.

↓

Scale hierarchy is created by the coupling between plasma flow and singular perturbation of the Hall effect in the equilibrium of Hall MHD.
Minimum energy state of Hall MHD

Minimum energy ($E$) with constraints of helicities ($H_1, H_2$) are characterized by the following ill-posed variational principle.


$$\delta (E - \mu_1 H_1 - \mu_2 H_2) = 0$$

$$E = \frac{1}{2} \int_{\Omega} (|B|^2 + |V|^2) \, dx,$$  \hspace{1cm} (4)

$$H_1 = \frac{1}{2} \int_{\Omega} A \cdot B \, dx,$$  \hspace{1cm} (5)

$$H_2 = \frac{1}{2} \int_{\Omega} (A + \varepsilon V) \cdot (B + \varepsilon \nabla \times V) \, dx.$$  \hspace{1cm} (6)
Euler-Lagrange equation

Double Beltrami field

\[ B = C_+ G_+ + C_- G_- , \quad V = (\lambda_+ - \mu_1)C_+ G_+ + (\lambda_- - \mu_1)C_- G_- , \]

where \( \nabla \times G_\pm = \lambda_\pm G_\pm \) and \( \lambda_\pm = \frac{1}{2} \left[ (\mu_2^{-1} + \mu_1) \pm \sqrt{(\mu_2^{-1} - \mu_1)^2 - 4} \right] . \)

⇒ Minimum energy state

One of \( \lambda_\pm \) approaches to \( \infty \) and corresponding \( C_\pm \) becomes 0.

One of Beltrami fields \( (G_\pm) \) becomes singular and vanish.

(Almost every where zero)

⇒ Taylor Relaxed State \( \nabla \times B = \lambda_0 B, \quad V = 0 . \)

(\( \lambda_0 \) is the minimum eigenvalue of the curl operator.)

In the dissipative system, the kinetic (flow) energy may concentrates in the small scale (singular part) and dissipates intensively.
**Working hypothesis**

Hall term leads to

- Generating the perpendicular flows
- Creation of small scale structures
- Strong kinetic energy dissipation

on the non-linear dynamics, too.

⇓

In order to verify the hypothesis, we compare the dynamics of relaxation process in the MHD and Hall-MHD systems by numerical simulations.
Simulation model

2-D Hall-MHD equations (Assuming $\partial_z = 0$)

Using $\psi(x, y)$ and $\phi(x, y)$, write $B$ and $V$ in the form of

$$B = \nabla \psi(x, y) \times \nabla z + B_z(x, y) \nabla z,$$

$$V = \nabla \phi(x, y) \times \nabla z + V_z(x, y) \nabla z.$$
The Hall term brings about the coupling between the poloidal \((\psi, \phi)\) and toroidal fields \((B_z, V_z)\).

As for 2-D equilibrium, \(\phi\) must be a certain function of \(\psi\) in MHD, however, \(\phi\) is no longer the function of \(\psi\) in Hall MHD, since \(\{\phi - \varepsilon B_z, \psi\} = 0\) in (9).

We solve numerically the Hall MHD equations (7) - (10) paying attention to perpendicular flows to the magnetic field (the crossing of \(\psi\) and \(\phi\)) and small scale structures.
• Domain $\Omega = [0, 1] \times [0, 1]$ with periodic boundary condition.
• $R_e = R_m = 10^4$.
• Domain is implemented on $200 \times 200$ point grids.
• Initial condition

$$\psi = [\sin(2\pi x) \sin(2\pi y) + 0.1 \sin(6\pi x) \sin(6\pi y)] / (2\pi),$$

(11)

$$\phi = \alpha_v \sin(2\pi x) \sin(2\pi y) / (2\pi),$$

(12)

$$B_z = V_z = 0.$$  

(13)

Simulations of MHD ($\varepsilon = 0$) and Hall MHD ($\varepsilon = 0.1$) are carried out for two initial conditions; sub-Alfvénic flow ($\alpha_v = 0.5$), and Alfvénic flow ($\alpha_v = 1.0$).

We observe (I) the energy and the energy spectrum, and (II) the perpendicular component of the poloidal flow to the poloidal magnetic field.
Numerical result

Sub-Alfvénic flow case

\( \psi \) and \(-\Delta \psi\) in the MHD model

\[
\begin{array}{ccc}
\psi \\
\hline
\text{t=5} & \text{t=10} & \text{t=100} \\
\Delta \psi \\
\end{array}
\]
ψ and −Δψ of the Hall-MHD model

Turbulence is developed and the fields relax to the large scale structure in both the MHD model and Hall MHD model.
The kinetic energy dissipates much faster and remains almost no-flow in the final state in the Hall MHD system.
The Hall term produces small scale structures effectively, since the energy density at high $k_x$ of Hall MHD is larger than that of MHD.
Time evolution of $< V_{p\perp}^2 >$ in the MHD and Hall-MHD models

Perpendicular flows are generated much more in Hall MHD than in MHD.
Alfvénic flow case
ψ and $-\Delta \psi$ in the MHD model

ψ

$-\Delta \psi$

t=10

t=50

t=100
$\psi$ and $-\Delta \psi$ in the Hall-MHD model

The fields governed by MHD evolve very slowly comparing with the fields done by Hall MHD.
Time evolution of energy in the MHD and Hall-MHD models

The Hall term leads the kinetic energy to dissipate much faster.
Energy spectrum $E(k_x)$ in the MHD and Hall-MHD models

The small scales are generated by the Hall term effectively.
Time evolution of $< V^2_{p\perp} >$ in the MHD and Hall-MHD models

Perpendicular flows are generated much more in Hall MHD than in MHD.
Summary


In the dynamics of the flowing plasmas, the singular perturbation of the Hall effect works mainly on

- Generation of perpendicular flow
- Creation of small scale structures
- Strong kinetic energy dissipation

which can be predicted by the analysis of equilibrium.

In Alfvénic flow case, the difference between MHD and Hall MHD becomes much clearer, that is, the singular perturbation of the Hall term may work much more. We can say that the non-equilibrium magnetic field is frozen in the Alfvénic equilibrium flow, and the fields are somewhat stabilized in the MHD model, however, it is not the case in the Hall MHD model because of the singular perturbation of the Hall effect.
Appendix “ill-posed variational principle”

Minimizer of the energy of a real function \( u(x) \) defined on \((0, \pi)\) such that \( u(0) = u(\pi) = 0; \) \( F_1(u) = \int_0^\pi u^2\,dx. \)

Without any constraint, the minimizer is the trivial \( u(x) = 0. \)

A “fragile” constraint is imposed; \( F_2(u) = \int_0^\pi \left(\frac{du}{dx}\right)^2\,dx = 1. \)

The variational principle \( \delta(F_1 - \nu F_2) = 0 \) yields \( -\frac{d^2}{dx^2} u = \frac{1}{\nu} u. \)

The solution must be one of the eigenfunctions

\[
 u(x) = C \sin(\alpha x) \quad (\alpha = \pm \sqrt{1/\nu} = \pm 1, \pm 2, \pm 3 \ldots).
\]

\( F_2 = 1 \) leads to \( C^2 = 2/(\pi \alpha^2) = 2\nu/\pi, \) and \( F_1(u) = 1/\alpha^2 = \nu. \)

The largest eigenvalue \( (\alpha^2 \to \infty) \) gives the minimum of \( F_1 \)

\( \inf F_1(u) = 0, \) i.e. \( u(x) = 0. \) The fragility of \( F_2(u) \) is due to the fact that it includes a higher-order derivative in comparison with the target functional \( F_1(u). \)
Time evolution of $E$, $H_1$ and $\hat{H}_2 = (H_2 - H_1)/\varepsilon$ (; cross helicity in MHD limit) in (a) MHD and (b) Hall MHD. If $H_1$ and/or $\hat{H}_2$ do not change much while $E$ diminishes, the “selective dissipation” of $E$ may yield the minimizer of $E$ under the constraints on the approximate constants of motion.