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The non-conservation of the conventional helicity is because vortex filaments are no longer pure states in relativistic dynamics.
Background I

vortex — common *structure* in the Universe

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† from http://www.astronomynotes.com/cosmolgy/s12.htm
vortex — common *structure* in the Universe

How was the first vortex created?

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For a *co-moving* loop $L(t)$, the rate of change of the circulation is

$$\frac{d}{dt} \int_{L(t)} P \cdot d\xi = \int_{L(t)} \left[ \partial_t P - v \times (\nabla \times P) \right] \cdot d\xi.$$
For a \textit{co-moving} loop $L(t)$, the rate of change of the circulation is

$$\frac{d}{dt} \int_{L(t)} P \cdot \ell d\xi = \int_{L(t)} [\partial_t P - v \times (\nabla \times P)] \cdot \ell d\xi.$$

In a \textit{barotropic fluid},

$$\partial_t P - v \times (\nabla \times P) = -\nabla (h + mv^2/2).$$

Thus, we obtain \textit{Kelvin’s circulation theorem}:

$$\frac{d}{dt} \int_{L(t)} P \cdot \ell d\xi = 0.$$
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Generalizing $\mathbf{P} = m\mathbf{v} + q\mathbf{A}$, we obtain the circulation theorem for the *canonical momentum*.
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Generalizing $P = mv + q\mathbf{A}$, we obtain the circulation theorem for the *canonical momentum*.

Neglecting the electron mass (MHD model), we obtain the circulation theorem for the magnetic field, i.e. *Alfvén’s theorem*.
For a general vector $\mathbf{v}$ and a covector (1-form) $P$, 

$$\frac{d}{dt} \int_{L(t)} P = \int_{L(t)} (\partial_t + L_{\mathbf{v}})P.$$
Background III (baroclinic effect)

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2. An ideal equation of motion may be written as 
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with some scalar $\varepsilon$ (representing the total enthalpy).
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Hence, a momentum (1-form) generated by a scalar (0-form) is naturally “exact”, posing a challenge of generating a vortex (2-form).

The space-time distortion by the relativistic effect, however, brings about a relativistic baroclinic effect, breaking the exactness of the thermal force:

$$(\partial_t + L_{\mathbf{v}})P = \gamma^{-1}d\varepsilon.$$

This mechanism can create a seed magnetic field (EM vorticity) in a cosmological plasma; [Mahajan-Yoshida, PRL 105 (2010), 095005]
The conventional helicity of \( \mathbf{b} = \nabla \times \mathbf{a} \) is (on a fixed \( \Omega \subseteq \mathbb{R}^3 \))

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C = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \, d^3x.
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In a fluid/plasma, we consider the canonical momentum $\mathbf{a} \leftarrow \mathbf{P} = m\mathbf{v} + q\mathbf{A}$ and its vorticity $\mathbf{b} \leftarrow \mathbf{\omega} = m\nabla \times \mathbf{v} + q\mathbf{B}$. 

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Geometrical Theory of Vortex  
2013/11/12  
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The conventional helicity of $b = \nabla \times a$ is (on a fixed $\Omega \subseteq \mathbb{R}^3$)

$$C = \int_{\Omega} a \cdot b \, d^3x. \tag{1}$$

In a fluid/plasma, we consider the canonical momentum $a \leftarrow P = mv + qA$ and its vorticity $b \leftarrow \omega = m\nabla \times v + qB$.

In a barotropic fluid, $P$ obeys

$$\partial_t P - v \times \omega = -\nabla \varepsilon, \tag{2}$$

where $\varepsilon = h + mv^2/2 + \phi$ ($h(\rho)$: enthalpy, $\phi$: E potential).
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Under a boundary condition \( \mathbf{n} \cdot \mathbf{b} = 0 \), \( C \) is conserved.
To delineate the topological meaning of $C$ in the simplest form, consider a pair of vortex filaments:

$$b d^3 x = \ell_1 d\xi_1 + \ell_2 d\xi_2,$$

where $\ell_1$ and $\ell_2$ are $\delta$-measures on loops $\Gamma_1$ and $\Gamma_2$.

By (generalized) Stokes' formula,

$$C = \int_{\mathbb{R}^3} a \cdot b d^3 x = \oint_{\Gamma_1} a \cdot \ell_1 d\xi_1 + \oint_{\Gamma_2} a \cdot \ell_2 d\xi_2 = 2\mathcal{L}(\Gamma_1, \Gamma_2). \quad (3)$$

By Biot-Savart integral, we may write

$$C = 2 \times \frac{1}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{(x_1 - x_2) \cdot \ell_1 d\xi_1 \times \ell_2 d\xi_2}{|x_1 - x_2|^3}. \quad (4)$$
We denote the Minkowski space-time by $\mathcal{M} \cong \mathbb{R}^4$; on a reference frame, we write

\[ x^\mu = (ct, x, y, z), \quad x_\mu = (ct, -x, -y, -z). \]

The space-time gradients are denoted by

\[ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{c \partial t}, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{c \partial t}, -\nabla \right). \]

The relativistic 4-velocity is defined by the proper-time derivative:

\[ U^\mu = \frac{dx^\mu}{ds} = (\gamma, \gamma v/c), \quad U_\mu = \frac{dx_\mu}{ds} = (\gamma, -\gamma v/c), \]

where $ds^2 = dx^\mu dx_\mu$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. 

\[ \]
Basic definitions of Minkowski space-time (2)

- The fluid 4-momentum is a 1-form $P = P_\mu dx^\mu \in T^* M$ with $P_\mu = (h/c) U_\mu$ ($h$ is the molar enthalpy, $h/c^2$ is the effective mass density).
- For a charged fluid (plasma), the canonical 4-momentum is $\mathcal{P} = P + qA$, where $q$ is the charge.
- The 4-velocity $U^\mu$ is a vector field $U = U^\mu \partial_\mu \in TM$, which generates a diffeomorphism $\mathcal{T}_U(s)$ by
  \[ \frac{\text{d}}{\text{d}s} \mathcal{T}_U(s) = U, \]  
  (5)
- The “$t$-plane cross-section” is, for a fixed parameter $t \in \mathbb{R}$,
  \[ \Xi(t) = \{(x^0, x^1, x^2, x^3); \; x^0 = ct, \; (x^1, x^2, x^3) \in X\}. \]  
  (6)
- The “proper time $s$-plane cross-section” is
  \[ \tilde{\Xi}(s) = \mathcal{T}_U(s)\Xi(0). \]  
  (7)
In terms of the canonical momentum $\mathcal{P} = P + qA$, the matter-EM field tensor is a 2-form

$$\mathcal{M} = d\mathcal{P} = \partial_\mu \mathcal{P}_\nu dx^\mu \wedge dx^\nu.$$ 

The equation of motion reads, assuming a barotropic relation $T dS = d\theta$,

$$i_U \mathcal{M} = -c^{-1} d\theta.$$  \hspace{1cm} (8)

Or, invoking the Lie derivative,

$$L_U \mathcal{P} = c^{-1} d(h + q\phi - \theta).$$  \hspace{1cm} (9)
Notation (EM analogy)

- The canonical momentum (or dressed EM potential) is
  \[ \mathcal{P}^\mu = (\mathcal{P}^0, \mathcal{P}) \equiv \mathcal{A}^\mu = (A^0, A). \] (10)

- EM vectors:
  \[ \mathcal{E} = -\nabla A^0 - (1/c)\partial_t A, \]
  \[ \mathcal{B} = \nabla \times A. \]

- Field tensor:
  \[ \mathcal{M}^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu = \begin{pmatrix}
    0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\
    \mathcal{E}_1 & 0 & -\mathcal{B}_3 & \mathcal{B}_2 \\
    \mathcal{E}_2 & \mathcal{B}_3 & 0 & -\mathcal{B}_1 \\
    \mathcal{E}_3 & -\mathcal{B}_2 & \mathcal{B}_1 & 0
  \end{pmatrix}. \] (11)
The helicity $C = \int_X \mathbf{a} \cdot \mathbf{b} \, d^3x$ is naturally generalized as

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The helicity $C = \int_X \mathbf{a} \cdot \mathbf{b} \, d^3x$ is naturally generalized as

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We find

$$\frac{d}{dt} C = c \int_X \mathbf{E} \cdot \mathbf{B} \, d^3x = 2 \int_X \gamma^{-1} \mathbf{B} \cdot \nabla \theta \, d^3x$$

$$= -2 \int_X \theta \mathbf{B} \cdot \nabla \gamma^{-1} \, d^3x, \quad (13)$$

showing that the relativistic factor $\gamma$ can break the constancy of the helicity.
We define a generalized helicity $\mathcal{C}$ in the Minkowski space-time by the integral of the 3-form $\mathcal{K} = \mathcal{P} \wedge d\mathcal{P}$ over a co-moving 3D spatial volume $V(s) = T_u(s)V_0 \ (V_0 \in \mathfrak{X}(0))$:

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\[
\mathcal{C}(s) = \int_{V(s)} \mathcal{P} \wedge d\mathcal{P}.
\]  

(14)

\textbf{Theorem}

\textit{The helicity $\mathcal{C}(s)$ is a constant of motion:}

\[
\frac{d}{ds} \mathcal{C}(s) = 0.
\]
A pair of geometric objects (chains) having co-dimension $\leq 2$ may link; for example, two loops may link in $\mathbb{R}^3$. 
What does the helicity conservation constrain?

1. A pair of geometric objects (chains) having co-dimension \( \leq 2 \) may link; for example, two loops may link in \( \mathbb{R}^3 \).

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4. Since the vorticity \( \mathcal{M} = dP \) is a 2-form, the helicity describes the link of 2D surfaces in 4D space-time.
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4. Since the vorticity \( \mathcal{M} = d\mathcal{P} \) is a 2-form, the helicity describes the link of 2D surfaces in 4D space-time.

5. The link of surfaces yields a topological constraint on loops (vortex-filaments) that are the \( s \)-plane (or \( t \)-plane) cross sections of the vorticity surfaces.
Let $M$ be a smooth manifold of dimension $n$, and $\Omega \subset M$ be a $p$-dimensional connected null-boundary submanifold of class $C^1$. Each $\Omega$ is equivalent to a pure-sate functional $\eta_\Omega$ on $\wedge^p T^* M$:

$$\eta_\Omega(\omega) = \int_\Omega \omega = \int_M \mathcal{J}(\Omega) \wedge \omega,$$

where $\mathcal{J}(\Omega) = \wedge^{n-p} \delta(x^\mu - \xi^\mu) dx^\mu$ is a $\delta$-measure supported on $\Omega$. We call $\mathcal{J}(\Omega)$ a pure state $(n - p)$-form, which is a member of the Hodge-dual space of $\wedge^p T^* M$. 

Definition (pure sate)

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A pure-state functional \( \eta_{\Sigma}(\omega) = \int_{\Sigma} \omega \) of vorticity 2-forms \( \omega \) is
\[
\mathcal{J}(\Sigma) = \delta_{\Sigma} \ast dy_1 \wedge dy_2 = \delta_{\Sigma} m, \quad m = \frac{1}{2} m_{\mu \nu} dx^{\mu} \wedge dx^{\nu},
\]
where \( \delta_{\Sigma} \) is the 2D \( \delta \)-function supported on a surface \( \Sigma \),

On the \( s \)-plane cross-section \( \tilde{\Gamma}(s) = \tilde{\Xi}(s) \cap \Sigma \), we obtain a pure-state "relativistic B-filament", which is a singular 3-form such that
\[
\mathcal{J}_b(\tilde{\Gamma}(s)) = \tilde{\rho}_b(s) \mathcal{J}(\Sigma) = -\delta_{\tilde{\Xi}(s)} \mathcal{U} \wedge \mathcal{J}(\Sigma),
\]
where \( \mathcal{U} = U_{\mu} dx^{\mu} \).
Consider a pair of disjoint loops $\tilde{\Gamma}_1(s)$ and $\tilde{\Gamma}_2(s)$, and their orbits $\Sigma_1 = \bigcup \tilde{\Gamma}_1(s)$ and $\Sigma_2 = \bigcup \tilde{\Gamma}_2(s)$.

On each $\Sigma_\ell$, we give a pure-state vorticity $M_\ell = \mathcal{J}(\Sigma_\ell)$.

Then, on each $\tilde{\Gamma}_\ell(s)$, we obtain a pure-sate $B$-filament $\mathcal{J}_b(\tilde{\Gamma}(s))$.

Denoting $M = M_1 + M_2$ and $P = P_1 + P_2 = FM_1 + FM_2$, the relativistic helicity of the twin vorticity evaluates as

$$\mathcal{C}(s) = \int_{V(s)} P \wedge M = \int_{\tilde{\Gamma}_1(s)} P_2 + \int_{\tilde{\Gamma}_2(s)} P_1.$$  \hspace{1cm} (17)
We denote $\delta = *d*$, and $\Box = \delta d + d\delta$ (d’Alembertian). We invert $\Box$ by the Liénard-Wiechert integral operator, which we denote by $\Box^{-1}$. We can define

$$\mathcal{P} = \mathcal{F}\mathcal{M} = \Box^{-1}\delta\mathcal{M}.$$
Theorem (link in Minkowski space-time)

Let $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ be a twin vortex generated by a pair $(\ell = 1, 2)$ of pure-state B-filaments $\mathcal{J}_b(\tilde{\Gamma}_1)$ and $\mathcal{J}_b(\tilde{\Gamma}_2)$.

1. The relativistic B-filaments $\mathcal{J}_b(\tilde{\Gamma}_\ell(s))$ continue to be pure states.
2. The relativistic helicity

\[
\mathcal{C}(s) = \int_{\tilde{\Gamma}_1(s)} \mathcal{F} \mathcal{M}_2 + \int_{\tilde{\Gamma}_2(s)} \mathcal{F} \mathcal{M}_1. \tag{19}
\]

is a constant of motion.

3. The constant $\mathcal{C}(s)/2$ is the linking number $\mathcal{L}(\tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s))$, which may be represented as (generalizing Gauss’ integral)

\[
\mathcal{L}(\tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s)) = \int \mathcal{F} \mathcal{M}_2 \wedge \mathcal{J}_b(\tilde{\Gamma}_1(s)) = \int \mathcal{F} \mathcal{M}_1 \wedge \mathcal{J}_b(\tilde{\Gamma}_2(s)). \tag{20}
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A relativistic helicity $C(s)$ has been formulated, which is conserved in a barotropic flow.
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2 The conservation of $\mathcal{C}(s)$ imposes a topological constraint on the relativistic $\mathbf{B}$-filaments in the Minkowski space-time.
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For a pair of pure-state $\mathbf{B}$-filaments, $\mathcal{C}$ measures their linking number.
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ZY, Y. Kawazura and T. Yokoyama, *Relativistic helicity and link in Minkowski space-time*; arXiv:1308.2455