Canonical Hamiltonian mechanics of Hall MHD and its limit to ideal MHD

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Macroscopic systems are often noncanonical, i.e. the phase space is foliated.

The Hamiltonian system of HMHD (or MHD), formulated in terms of Eulerian variables, is noncanonical.

We formulate canonized Hamiltonian systems of HMHD which have a hierarchized set of canonical variables; in the simplest system, the ion vorticity and magnetic field have integral surfaces.

Renormalizing the singularity scaled by the reciprocal Hall parameter (as the ion vorticity surfaces and the magnetic surfaces are set to merge), we delineate the singular limit to ideal MHD.

A Hamiltonian system is formulated on a *phase space* $V$:

$$\frac{d}{dt} u = \mathcal{J}(u) \partial_u H(u).$$

The adjoint representation is

$$\frac{d}{dt} F(u) = [F(u), H(u)], \quad [F, H] = \langle \partial_u F, \mathcal{J} \partial_u H \rangle.$$
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Separating the center (or the Casimir elements) of the Poisson algebra, one may unearth symplectic leaves on which the local dynamics (coadjoint orbits) has a canonical structure:

$$\mathcal{J} = \mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$
The governing equations are

\[ \begin{align*}
\partial_t \rho &= -\nabla \cdot (\mathbf{V} \rho), \\
\partial_t \mathbf{V} &= -\left(\nabla \times \mathbf{V}\right) \times \mathbf{V} - \nabla \left(h + \frac{V^2}{2}\right) + \rho^{-1} \left(\nabla \times \mathbf{B}\right) \times \mathbf{B}, \\
\partial_t \mathbf{B} &= \nabla \times (\mathbf{V} \times \mathbf{B}).
\end{align*} \]

Boundary conditions are \( n \cdot \mathbf{V} |_{\partial \Omega} = 0 \) and \( n \cdot \mathbf{B} |_{\partial \Omega} = 0 \).

The latter is a consequence of \( n \times \mathbf{E} |_{\partial \Omega} = 0 \).

The MHD equations can be cast into a Hamiltonian form with \( u = (\rho, \mathbf{V}, \mathbf{B}) \) and

\[ \begin{align*}
H &= \int_{\Omega} \left\{ \rho \left[ \frac{V^2}{2} + \mathcal{E}(\rho) \right] + \frac{B^2}{2} \right\} \, d^3x, \\
\mathcal{J} &= \begin{pmatrix}
0 & -\nabla \cdot \\
-\nabla & -\rho^{-1} \left(\nabla \times \mathbf{V}\right) \times \\
0 & \nabla \times (\circ \times \rho^{-1} \mathbf{B})
\end{pmatrix} \begin{pmatrix}
0 \\
\rho^{-1} \left(\nabla \times \circ \times \mathbf{B}\right) \times \\
0
\end{pmatrix}. \quad (1)
\end{align*} \]
The Poisson operator $\mathcal{J}(u)$ of MHD has, at least, the following Casimir elements:

$$C_1 = \frac{1}{2} \int_{\Omega} A \cdot B \, d^3x,$$

$$C_2 = \frac{1}{2} \int_{\Omega} V \cdot B \, d^3x,$$

$$C_3 = \int_{\Omega} \rho \, d^3x,$$

where $A$ is the vector potential ($A = \text{curl}^{-1}B$), which is evaluated with a fixed gauge and boundary conditions.

When the domain $\Omega$ is multiply connected, the fluxes

$$\Phi_\ell = \int_{\Sigma_\ell} n \cdot B \, d^3x = \int_{\Omega} \sigma \wedge B$$

are (singular) Casimir elements.
The HMHD equations are
\[
\partial_t \rho + \nabla \cdot (\mathbf{V} \rho) = 0, \\
\partial_t \mathbf{V} - \mathbf{V} \times (\nabla \times \mathbf{V}) + \rho^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla (h + V^2/2), \\
\partial_t \mathbf{B} - \nabla \times [(\mathbf{V} - \epsilon \rho^{-1} \nabla \times \mathbf{B}) \times \mathbf{B}] = 0.
\]

We assume the same boundary conditions, as well as
\[
n \cdot (\nabla \times \mathbf{B})|_{\partial \Omega} = 0, \tag{6}
\]
which is necessary for the Poisson bracket.

We modify the Poisson operator (2) as
\[
\mathcal{J} = \begin{pmatrix}
0 & -\nabla \cdot \\
-\nabla & -\rho^{-1}(\nabla \times \mathbf{V}) \times \\
0 & \nabla \times (\mathbf{o} \times \rho^{-1} \mathbf{B}) \\
0 & -\epsilon \nabla \times [\rho^{-1}(\nabla \times \mathbf{o}) \times \mathbf{B}]
\end{pmatrix}. \tag{7}
\]

Using the same state vector \( u \equiv (\rho, \mathbf{V}, \mathbf{B}) \) and the Hamiltonian (1), Hamilton’s equation gives the HMHD equations.
Invoking five scalar functions (Clebsch potentials) $\varphi_0, \varphi_1, \varphi_2, \sigma,$ and $\psi,$ we parameterize

$$P = -\nabla \varphi_0 - \sigma \nabla \varphi_1,$$

$$A = \psi \nabla \varphi_2,$$

Here we assume that all Clebsch potentials are single-valued smooth functions.

We assume boundary conditions

$$\mathbf{n} \cdot \nabla \varphi_0 |_{\Gamma_i} = 0, \quad \mathbf{n} \cdot \nabla \varphi_1 |_{\Gamma_i} = 0, \quad \mathbf{n} \cdot \nabla \varphi_2 |_{\Gamma_i} = 0,$$

$$\psi |_{\Gamma_i} = c_i, \quad \sigma |_{\Gamma_i} = d_i,$$

where $\partial \Omega = \bigcup \Gamma_i,$ $c_i$ and $d_i$ ($i = 1, \cdots, m$) are real constants.
Operating by $\nabla \times$, we obtain
\begin{align*}
\Omega &= -\nabla \sigma \times \nabla \varphi_1, \\
B &= \nabla \psi \times \nabla \varphi_2.
\end{align*}
(10) (11)

Evidently, $\Omega \cdot \nabla \sigma = \Omega \cdot \nabla \varphi_1 = 0$, and $B \cdot \nabla \psi = B \cdot \nabla \varphi_2 = 0$, implying that the vortex lines and magnetic field lines are integrable.

Clebsch parameterized submanifold is zero-helicity:
\[ A \cdot (\nabla \times A) = 0, \]

thus $C_1 = 0$, and
\begin{align*}
C'_2 &= \int_{\Omega} P \cdot (\nabla \times P) \, d^3x \\
&= \int_{\Omega} \nabla \cdot (\varphi_0 \nabla \sigma \times \nabla \varphi_1) \, d^3x = -\int_{\partial \Omega} \varphi_0 n \cdot \Omega \, d^2x = 0.
\end{align*}
In terms of the Clebsch variables $u = (\varphi_0, \rho, \varphi_1, \mu_1, \varphi_2, \mu_2)$, where $\sigma = \mu_1/\rho$ and $\psi = \mu_2/\rho$, we consider a Lagrangian

$$L = \int_\Omega \left[ (\rho \dot{\varphi}_0 + \mu_1 \dot{\varphi}_1 + \epsilon^{-1} \mu_2 \dot{\varphi}_2) - H(u) \right] \, d^3x,$$

(12)

with the Clebsch-parameterized Hamiltonian $H = \int_\Omega H \, d^3x$, where

$$H(u) = \rho \left( \frac{|\mathbf{P} - \epsilon^{-1} \mathbf{A}|^2}{2} - \mathcal{E}(\rho) \right) - \frac{|
abla \times \mathbf{A}|^2}{2}$$

$$= \rho \left( \frac{|
abla \varphi_0 + (\mu_1/\rho) \nabla \varphi_1 + \epsilon^{-1}(\mu_2/\rho) \nabla \varphi_2|^2}{2} + \mathcal{E}(\rho) \right)$$

$$+ \frac{|
abla (\mu_2/\rho) \times \nabla \varphi_2|^2}{2}. \quad (13)$$
The variation of the action with respect to the canonical variables yields a system of canonical equations:

\[
\begin{align*}
\dot{\varphi}_0 &= \partial_\rho H = -(\mathbf{V} \cdot \nabla)\varphi_0 + h - V^2/2 - \rho^{-1} \mathbf{J} \cdot \mathbf{A}, \\
-\dot{\rho} &= \partial_{\varphi_0} H = \nabla \cdot (\mathbf{V} \rho), \\
\dot{\varphi}_1 &= \partial_{\mu_1} H = -(\mathbf{V} \cdot \nabla)\varphi_1, \\
-\mu_1 &= \partial_{\varphi_1} H = \nabla \cdot (\mathbf{V} \mu_1), \\
\dot{\varphi}_2 &= \epsilon \partial_{\mu_2} H = -(\mathbf{V} \cdot \nabla)\varphi_2 + \epsilon (\rho^{-1} \mathbf{J} \cdot \nabla)\varphi_2 = -(\mathbf{V}_e \cdot \nabla)\varphi_2, \\
-\mu_2 &= \epsilon \partial_{\varphi_2} H = \nabla \cdot (\mathbf{V} \mu_2) - \epsilon \nabla \cdot (\rho^{-1} \mathbf{J} \mu_2) = \nabla \cdot (\mathbf{V}_e \mu_2). 
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\dot{\varphi}_2 &= \epsilon \partial_{\mu_2} H = -(\mathbf{V} \cdot \nabla)\varphi_2 + \epsilon (\rho^{-1} \mathbf{J} \cdot \nabla)\varphi_2 = -(\mathbf{V}_e \cdot \nabla)\varphi_2, \\
-\dot{\mu}_2 &= \epsilon \partial_{\varphi_2} H = \nabla \cdot (\mathbf{V} \mu_2) - \epsilon \nabla \cdot (\rho^{-1} \mathbf{J} \mu_2) = \nabla \cdot (\mathbf{V}_e \mu_2).
\end{align*}
\]

After a bit lengthy calculation, we find that \( \mathbf{P} = -\nabla \varphi_0 - \sigma \nabla \varphi_1 \) and \( \mathbf{A} = \psi \nabla \varphi_2 \) satisfies the HMHD equations.
In order to represent general dynamics, we put

\[ \mathbf{P} = -\nabla \varphi_0 - \sigma \nabla \varphi_1 - \sigma' \nabla \varphi_1' - \sigma'' \nabla \varphi_1'', \quad (14) \]
\[ \mathbf{A} = \psi \nabla \varphi_2 + \psi' \nabla \varphi_2' + \psi'' \nabla \varphi_2''. \quad (15) \]

In order to represent general dynamics, we put

\[ \mathbf{P} = -\nabla \varphi_0 - \sigma \nabla \varphi_1 - \sigma' \nabla \varphi'_1 - \sigma'' \nabla \varphi''_1, \]

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Extending the canonical variables as

\[ u = (\varphi_0, \rho, \varphi_1, \mu_1, \varphi'_1, \mu'_1, \varphi''_1, \mu''_1, \varphi_2, \mu_2, \varphi'_2, \mu'_2, \varphi''_2, \mu''_2). \]

we consider a Lagrangian and a Hamiltonian such as

\[ L = \int_{\Omega} \left[ \left( \rho \dot{\varphi}_0 + \sum_{\ell} \mu_1^\ell \dot{\varphi}_1^\ell + \epsilon^{-1} \sum_{\ell} \mu_2^\ell \dot{\varphi}_2^\ell \right) - \mathcal{H}(u) \right] d^3x, \]

\[ \mathcal{H}(u) = \rho \left[ \frac{1}{2} \left| \nabla \varphi_0 + \sum_{\ell} \left( \mu_1^\ell / \rho \right) \nabla \varphi_1^\ell + \epsilon^{-1} \sum_{\ell} \left( \mu_2^\ell / \rho \right) \nabla \varphi_2^\ell \right|^2 + \mathcal{E}(\rho) \right] + \frac{1}{2} \sum_{\ell} \left| \nabla \left( \mu_2^\ell / \rho \right) \times \nabla \varphi_2^\ell \right|^2. \]
Suppose that $\mathbf{B}$ has fluxes

$$\Phi_j = \int_{\Sigma_j} \mathbf{n} \cdot \mathbf{B} \, d^2x \quad (j = 1, \ldots, \nu).$$
A topological issue: multiply-connected domain
Multiple-valued Clebsch potential

- Suppose that $\mathbf{B}$ has fluxes

$$\Phi_j = \int_{\Sigma_j} \mathbf{n} \cdot \mathbf{B} \, d^2x \quad (j = 1, \cdots, \nu).$$

- Then, we put

$$\tilde{\varphi}_2 = \varphi_2 - \sum_{j=1}^{\nu} \frac{\Phi_j}{2\pi c_i} \theta_j,$$

where $\varphi_2$ is a single-valued component, $\oint_{\partial \Sigma_j} d\theta_j = 2\pi$ for each loop $\partial \Sigma_j \subset \Gamma_i$ $(j = 1, \cdots, \nu)$, and $c_i = \psi|_{\Gamma_i}$. We have to assume that $\psi \nabla \tilde{\varphi}_2$ can be evaluated as a continuously differentiable function in $\Omega$, i.e. $\psi \ (= \mu_2/\rho)$ vanishes rapidly in the neighborhood of the phase singularities of $\tilde{\varphi}_2$. 
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The canonical variables are $u = (\varphi_0, \rho, \varphi_1, \mu_1, \tilde{\varphi}_2, \mu_2)$. We may use the same form of the Lagrangian to obtain canonical equations.
The HMHD Lagrangian is singular in the limit of $\epsilon \to 0$:

$$L = \int_{\Omega} \left[ \left( \rho \dot{\varphi}_0 + \mu_1 \dot{\varphi}_1 + \epsilon^{-1} \mu_2 \dot{\varphi}_2 \right) - \mathcal{H} \right] \, d^3 x,$$

$$\mathcal{H} = \rho \left( \frac{|\nabla \varphi_0 + (\mu_1/\rho) \nabla \varphi_1 + \epsilon^{-1} (\mu_2/\rho) \nabla \varphi_2|^2}{2} + \mathcal{E}(\rho) \right) + \frac{|\nabla (\mu_2/\rho) \times \nabla \varphi_2|^2}{2}.$$
The MHD limit

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- Introducing a new set of variables, we can rewrite the Lagrangian in a form that is amenable to the limit $\epsilon \to 0$; setting $\epsilon = 0$, we obtain a Lagrangian that yields the MHD equations.
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- We put

$$\begin{cases} 
\mu_1 = -\epsilon^{-1} \mu_2 + \delta, \\
\varphi_1 = \varphi_2 - \epsilon \gamma.
\end{cases} \quad (17)$$
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  \]
  \[
  \mathcal{H} = \rho \left( \frac{|\nabla \varphi_0 + (\mu_1/\rho)\nabla \varphi_1 + \epsilon^{-1}(\mu_2/\rho)\nabla \varphi_2|^2 + \mathcal{E}(\rho)}{2} + \frac{|\nabla (\mu_2/\rho) \times \nabla \varphi_2|^2}{2} \right).
  \]

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  \tag{17}
  \]

- Notice that the singular factor $\epsilon^{-1} \mu_2$ in the Lagrangian (12) is set to be canceled by $\mu_1$. 

The MHD limit

- The HMHD Lagrangian is singular in the limit of $\epsilon \to 0$:

$$L = \int_{\Omega} \left[ \left( \rho \dot{\varphi}_0 + \mu_1 \dot{\varphi}_1 + \epsilon^{-1} \mu_2 \dot{\varphi}_2 \right) - \mathcal{H} \right] \, d^3 x,$$

$$\mathcal{H} = \rho \left( \frac{\left| \nabla \varphi_0 + (\mu_1/\rho) \nabla \varphi_1 + \epsilon^{-1}(\mu_2/\rho) \nabla \varphi_2 \right|^2}{2} + \mathcal{E}(\rho) \right) + \frac{\left| \nabla \left( \mu_2/\rho \right) \times \nabla \varphi_2 \right|^2}{2}.$$

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- Notice that the singular factor $\epsilon^{-1} \mu_2$ in the Lagrangian (12) is set to be canceled by $\mu_1$.

- Notice that as $\epsilon \to 0$, $\psi + \epsilon \sigma = (\mu_2 + \epsilon \mu_1)/\rho \to 0$ and $\varphi_2 - \varphi_1 \to 0$, that is, the magnetic ($\mathbf{B} = \nabla \psi \times \nabla \varphi_2$) and canonical vortex ($\mathbf{\Omega} = \nabla \times \mathbf{P} = -\nabla \sigma \times \nabla \varphi_2$) surfaces coincide as $\epsilon \to 0$. 

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Canonized HMHD and MHD  
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Replacing the canonical pair variables $\varphi_1$ and $\mu_1$ by $\gamma$ and $\delta$, we invoke a new system of canonical variables

$$u' = (\varphi_0, \rho, \gamma, \mu_2, \varphi_2, \delta).$$

(18)

The Lagrangian is rewritten as

$$L = \int_{\Omega} \left[ \rho \dot{\varphi}_0 + \mu_2 \dot{\gamma} + \delta (\dot{\varphi}_2 - \epsilon \dot{\gamma}) - \rho \left( \frac{|\nabla \varphi_0 + (\mu_2 / \rho) \nabla \gamma + (\delta / \rho) \nabla (\varphi_2 - \epsilon \gamma)|^2}{2} + E(\rho) \right) - \frac{|\nabla (\mu_2 / \rho) \times \nabla \varphi_2|^2}{2} \right] \, d^3 x,$$

(19)

which is amenable to the limit of $\epsilon \to 0$.

Notice that the original canonical-pair variables $\mu_2$ and $\varphi_2$, parameterizing the magnetic field as $\mathbf{B} = \nabla (\mu_2 / \rho) \times \nabla \varphi_2$, no longer constitute a pair; each of them pairs with the new parameter, $\gamma$ or $\delta$. 

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Canonized HMHD and MHD 
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The MHD Lagrangian

The $\epsilon = 0$ canonical Lagrangian

$$L_{\text{MHD}} = \int_{\Omega} \left[ \rho \dot{\varphi}_0 + \mu_2 \dot{\gamma} + \delta \dot{\varphi}_2 ight.$$

$$+ \rho \left( \frac{|\nabla \varphi_0 + (\mu_2/\rho) \nabla \gamma + (\delta/\rho) \nabla \varphi_2|^2}{2} + \mathcal{E}(\rho) \right)$$

$$\left. - \frac{|\nabla (\mu_2/\rho) \times \nabla \varphi_2|^2}{2} \right] \, d^3 x,$$

(20)

... turns out to be a Lagrangian producing the MHD system.
The MHD Lagrangian

- The $\epsilon = 0$ canonical Lagrangian

\[
L_{\text{MHD}} = \int_{\Omega} \left[ \rho \dot{\varphi}_0 + \mu_2 \dot{\gamma} + \delta \dot{\varphi}_2 
- \rho \left( \frac{|\nabla \varphi_0 + (\mu_2/\rho) \nabla \gamma + (\delta/\rho) \nabla \varphi_2|^2}{2} + E(\rho) \right) 
- \frac{|\nabla (\mu_2/\rho \times \nabla \varphi_2)^2|}{2} \right] d^3x, \quad (20)
\]

turns out to be a Lagrangian producing the MHD system.

- The Euler-Lagrange equations are

\[
\begin{align*}
\dot{\varphi}_0 &= -(\mathbf{V} \cdot \nabla) \varphi_0 + h - \mathbf{V}^2/2 - \rho^{-1} \mathbf{J} \cdot \mathbf{A}, \\
\dot{\rho} &= \nabla \cdot (\mathbf{V} \rho), \\
\dot{\gamma} &= -(\mathbf{V} \cdot \nabla) \gamma + \rho^{-1} \mathbf{J} \cdot \nabla \varphi_2, \\
\dot{\mu}_2 &= \nabla \cdot (\mathbf{V} \mu_2), \\
\dot{\varphi}_2 &= -(\mathbf{V} \cdot \nabla) \varphi_2, \\
\dot{\delta} &= \nabla \cdot (\mathbf{V} \delta) - \mathbf{J} \cdot \nabla (\mu_2/\rho).
\end{align*}
\]
Introducing a core Lagrangian, we have formulated a canonical Hamiltonian system that describes dynamics on a symplectic leaf (co-adjoint orbit) of HMHD.
1 Introducing a core Lagrangian, we have formulated a canonical Hamiltonian system that describes dynamics on a symplectic leaf (co-adjoint orbit) of HMHD.

2 We have delineated the path to the MHD limit; to suppress the singularity scaled by $\epsilon^{-1}$, the two canonical pairs, parameterizing $\mathbf{P}$ and $\mathbf{A}$, are merged at order unity, and the original canonical-pair variables are split into separate canonical pairs coupling with new perturbative variables.
Introducing a core Lagrangian, we have formulated a canonical Hamiltonian system that describes dynamics on a symplectic leaf (co-adjoint orbit) of HMHD.

We have delineated the path to the MHD limit; to suppress the singularity scaled by $\epsilon^{-1}$, the two canonical pairs, parameterizing $P$ and $A$, are merged at order unity, and the original canonical-pair variables are split into separate canonical pairs coupling with new perturbative variables.

This is an interesting example of singular limits, exhibiting the advantage of the Hamiltonian formalism that relativizes coordinate and momentum variables.